Mathematics 201 Calculus of Several Variables

Professor Peter M. Higgins

June 27, 2017

This is the first of our Second Year level modules and it builds on the two previous calculus modules MA103 and MA107. We begin with *multiple integration* and the change of order in *iterated integrals*, which requires great care to be taken to identify the new limits of integration, especially of the inner integrals whose limits will depend on the variables remaining in the outer integrals. We then move on to change of variables, which involves the *Jacobian matrix*, with particular emphasis on *polar, cylindrical,* and *spherical coordinates.* In Set 3 we introduce the notion of *curvature*, which is a fundamental idea intrinsic to the curve being examined and is not dependent on the co-ordinate system used in its calculation. Throughout we will be using *parametrization of curves*, with *parametrization by arc length* of fundamental importance as with this parametrization the tangent vector at each point is *normalised* in that it is always of unit length.

In Set 4 we introduce the so-called big O and little o notation, which is often used in describing the behaviour of complex functions in terms of simpler ones together with terms that collectively vanish as we approach a particular limit or infinity.

In Set 5 we introduce *Leibniz's Rule* for differentiating through an intergal of a function of several variables and apply the idea to evaluate some integrals that are difficult to analyse just using single variable techniques.

Sets 6 and 7 introduce some special functions, they being the *gamma* and *beta functions*, which are both defined by integrals involving a parameter. In Set 7 we meet the important *Chebyshev polynomials.*

In set 8 we introduce the *Laplace transform*, an integral transform that can be used to solve differential equations while Set 9 introduces *Fourier series* and applies them in order to sum certain interesting and special series.

Finally Set 10 introduces *line integrals*, which are integrals defined along curves, for both real-valued functions (*scalar fields*) and for *vector fields.* This problem set leads naturally on to MA203, *Vector Calculus*, where this topic is taken further.

As always, all our problem sets are self-contained. The topics practised however may be explored further by searching the internet based on the italicised key words and phrases.

Problem Set 1 Multiple integrals

For Questions 1-5 evaluate the given double integrals.

1.

$$\int_{y=1}^{2}\int_{x=0}^{1} xy(x+y)\,dxdy.$$

2.

$$\int_{0}^{\frac{\pi}{4}}\int_{0}^{1} 2xy\sin y\,dxdy.$$

By changing the order of integration, evaluate the integrals in Question 3-5.

3.

$$\int_{x=0}^{1}\int_{y=\sqrt{x}}^{1} e^{y^3}\,dydx.$$

4.

$$\int_{0}^{\frac{\pi}{2}}\int_{x}^{\frac{\pi}{2}} \frac{\sin y}{y}\,dydx.$$

5.

$$\int_{0}^{y=1}\int_{0}^{x=\frac{y}{2}} e^{x^2}\,dxdy$$

In Questions 6 and 7, evaluate the given triple integrals.

6.

$$\int_{P} xze^{xy}\,dxdydz$$

where P is the cuboid $P=\{(x,y,z) : 0\le x\le 1,\, 0\le y\le 1,\, 0\le z\le 2\}$.

7.

$$\int_{0}^{\frac{\pi}{2}}\int_{0}^{1}\int_{0}^{\sqrt{1-x^2}} x\cos z\,dydxdz.$$

8. Find the volume in the first quadrant bounded by the plane $x+y+z=1$ by expressing the volume of a triple integral of the constant function 1. Also find the answer by treating the shape as a triangular pyramid.

The area S of the surface defined by the equation $z=f(x,y)$ that lies above the region R in the xy-plane is given by the integral

$$S=\int\int_{R}\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}dxdy.$$

9. Find the area of the portion of the cylinder $x^2+z^2=4$ lying above the rectangle defined by $0\le x\le 1$ and $0\le y\le 4$.

10. Find the surface area of the portion of the paraboloid $2z=x^2+y^2$ that is inside the cylinder $x^2+y^2=8$.

Problem Set 2 Change of variable in multiple integrals

Let $(u, v) \to (x(u, v), y(u, v))$ map S in the uv-plane in a one-to-one manner onto D is the xy-plane. Then

$$\int\int_D f(x, y)\, dxdy = \int\int_S f(x(u, v), y(u, v))|J|\, dudv$$

where the *Jacobian* J is given by

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}.$$

This extends to functions of more than two variables in the natural way.

1. Find $|J|$ for the change of variable to polar co-ordinates $x = r\cos\theta$, $y = r\sin\theta$ $(0 \le \theta < 2\pi)$ and more generally for *cylindrical co-ordinates* which include the third independent variable z.

2. Use a double polar integral to find the area enclosed by the 3-leafed rose $r = \sin 3\theta$.

3. By changing to polar co-ordinates, find $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}dxdy$.

4. Use Question 3 to deduce the value of $\int_{-\infty}^{\infty} e^{-x^2}dx$.

5. Use the result of Question 4 to show that the function $f(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}$ taken over the entire real line is a probability density function.

6. By converting to polar coordinates evaluate

$$\int_{-1}^{1}\int_{0}^{\sqrt{1-(x-1)^2}} \frac{x+y}{x^2+y^2}\, dxdu$$

7. Find the area of the *cardiod* $r = 1+\sin\theta$ by evaluating through evaluating the integral $A = \int_{\theta=0}^{2\pi}\int_{r=0}^{r=1+\sin\theta} rdrd\theta$..

8. Find the volume V of the *conical paraboloid* $z+x^2+y^2 = 4$ that lies above the xy-plane by expressing V as a triple integral in cylindrical co-ordinates.

Spherical co-ordinates $P(r, \theta, \phi)$ of $P(x, y, z)$ are given by $r = \sqrt{x^2 + y^2 + z^2}$, $x = r\cos\theta\sin\phi$, $y = r\sin\theta\sin\phi$ and $z = r\cos\phi$ where $0 \le \theta < 2\pi$ is the *polar angle* in the xy-plane formed by the projection of OP onto the horizontal xy-plane and $0 \le \phi \le \pi$ is the angle between the vertical z-axis and the ray OP.

9. Show that for the transformation from cartesian to spherical coordinates we have that for the 3×3 Jacobian J we have $J = r^2\sin\phi$.

10. Suppose a sphere of radius a has variable density $\rho = \rho_0(1 - \frac{r}{a})$, where ρ_0 is a constant. Find total mass M of the sphere by expressing M as a triple integral in spherical coordinates of the density over the volume of the sphere.

Problem Set 3 Plane Curvature

Let a curve C be parametrized as $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$ $(a \leq t \leq b)$ and let $\phi(t)$ be the angle that the tangent vector $\dot{\mathbf{r}}(t)$ makes with the x-axis.

1. Use the equation $\tan(\phi(t)) = \frac{\dot{y}(t)}{\dot{x}(t)}$ to show that

$$\frac{d\phi}{dt} = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}^2 + \dot{y}^2}.$$

2. Recall the *arc length* of C at time t is given by

$$s(t) = \int_a^t \sqrt{\dot{x}^2(u) + \dot{y}^2(u)}\, du = \int_a^t ||\dot{\mathbf{r}}(u)||\, du.$$

The *curvature* κ of C is defined as $\frac{d\phi}{ds}$, while the *radius of curvature* is defined as $\phi = \frac{1}{\kappa}$. Use Question 1 and the Chain rule to show that

$$\kappa = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{(\dot{x}^2 + \dot{y}^2)^{\frac{3}{2}}}.$$

3. Parametrize the circle of radius a centred at the origin and find its curvature and radius of curvature.

4. Find the curvature of the parabola $y = x^2$.

5. Find the curvature of the graph of the function $y = \ln(\cos x)$, $(-\frac{\pi}{2} \leq x \leq \frac{\pi}{2})$.

6. A *cycloid* is the path traced out by a point on the rim of a wheel of a moving car. A typical parametrization of a cycloid is $x = a(t - \sin t)$, $y = a(1 - \cos t)$. Find arc length of the portion of this cycloid corresponding to the time interval $[0, \alpha]$ and the length of one complete arch.

7. Find the radius of curvature ρ for a point on the cycloid of Question 6 corresponding to a given time $t = \alpha$.

8. Find the area under one arch of the cycloid of Question 6.

9. The *Cornu spiral* is parametrized by the equations

$$x(t) = \int_0^t \cos(u^2)du,\ y(t) = \int_0^t \sin(u^2)du.$$

Find the length of the spiral between the parameter values of 0 and t_0.

10. Show that the curvature of the Cornu spiral is $2t$ and deduce that the curve has a constant rate of change of curvature.

Problem Set 4 Big O and little o notation

We say that a function $f = O(g)$ for another function g taking only positive values if there exists a constant K such that $|f(x)| \leq Kg(x)$ for all x. Similarly if $\frac{f}{g} \to 0$ (as $x \to \infty$) we write $f = o(g)$ (as $x \to \infty$). We write $f = o(g)$ as $x \to a$ if

$$\lim_{x \to a} \frac{f(x)}{g(x)} = 0.$$

We write $f \sim lg$ if $\frac{f}{g} \to l$, (as $x \to \infty$ or as $x \to a$ as the case may be) where l is a non-zero constant. We also simply write $O(g)$ and $o(g)$ to denote an unspecified function f such that $f = O(g)$ or $f = o(g)$ as the case may be. Verify each of the following statements.

1. $f = O(1)$ if and only if f is a bounded function; $f = o(1)$ if and only if $f(x) \to 0$ as $x \to \infty$.
2. $x^m = o(x^{m+1})$.
3. $\operatorname{cosec}(x) - \cot x \sim \frac{1}{2}x$ as $x \to 0$.
4. $O(f) + O(g) = O(f + g)$.
5. $O(fg) = O(f)O(g)$. (Note that fg here denotes *function product* (and not function composition) so that $(fg)(x) = f(x)g(x)$.)
6. $O(f)o(g) = o(fg)$.
7. If $f \sim g$ then $f + o(g) \sim g$.
8. Suppose that $f(x)$ has a convergent Taylor series in some neighbourhood of x_0 so that

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

Show that

$$f(x - x_0) - a_0 - a_1(x - x_0) = o(x - x_0) \text{ as } x \to x_0.$$

9. Continuing Question 8, show that $l(x) = a_0 + a_1(x - x_0)$ is the only linear function such that $f(x) - l(x) = o(x - x_0)$ as $x \to x_0$.
10. Simplify

$$(n + O(n^{\frac{1}{2}})(n + O(\log n))^2$$

as $n \to \infty$.

Problem Set 5 Differentiation under the integral

1. *Leibniz's Rule* Assume that we may write $f(x+\Delta x,t) = f(x,t) + \frac{\partial f(x,t)}{\partial x}\Delta x + o(\Delta x)$ so that the remainder term has the form $o(\Delta x)$ independently of t. Show from first principles that the derivative of

$$g(x) = \int_a^b f(x,t)\,dt$$

is given by

$$g'(x) = \int_a^b \frac{\partial f(x,t)}{\partial x}\,dt.$$

The *Feynman Integration Trick*, which featured on an episode of 'Big Bang Theory' is illustrated in the following questions.

2. Let $f(b) = \int_0^1 \frac{x^b-1}{\log x}\,dx$. By differentiating under the integral sign, find $f'(b)$.

3. Integrate $f'(b)$ to recover an expression for $f(b)$ up to an integration constant C. Find the integration constant C by putting $b=0$.

4. Hence find $\int_0^1 \frac{x^2-1}{\log x}\,dx$.

5. By n-fold differentiation of the function

$$g(x) = \int_0^\infty e^{-tx}\,dt$$

show that

$$\int_0^\infty x^n e^{-x}\,dx = n!$$

6. This time put

$$f(b) = \int_0^\infty \frac{\sin x}{x} e^{-bx}\,dx$$

and compute $f'(b)$.

7. Integrate $f'(b)$ to recover $f(b)$ up to an integration constant C and find C by putting $b=\infty$.

8. Hence find

$$\int_{-\infty}^\infty \frac{\sin x}{x}\,dx.$$

9. Show that for $x \geq 0$ that $\cos x \geq 1 - \frac{x^2}{2}$.

10. Does

$$\int_0^\infty \frac{\cos x}{x}\,dx$$

exist?

Problem Set 6 Special functions I: The Gamma and Beta functions

The *gamma function* is defined by $\Gamma(t) = \int_0^\infty x^{t-1}e^{-x}\,dx$.

1. By integrating by parts show that $\Gamma(t+1) = t\Gamma(t)$ and hence deduce that $\Gamma(n+1) = n!$ for any non-negative integer n.
2. Show that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.
3. Show that

$$\Gamma(\frac{1}{2}+n) = \frac{(2n)!}{4^n n!}\sqrt{\pi}.$$

4. Show that

$$\Gamma(\frac{1}{2}-n) = \frac{(-4)^n n!}{(2n)!}\sqrt{\pi}.$$

The beta function is defined for positive real number pairs (x, y) as

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt.$$

5. Write

$$\Gamma(x)\Gamma(y) = (\int_0^\infty e^{-u}u^{x-1}\,du)(\int_0^\infty e^{-v}v^{y-1}\,dv)$$

as one double integral.

6. For the change of variables $u = zt$ and $v = z(1-t)$ find $|J(z,t)|$ where J is the Jacobian of the transformation, and the corresponding ranges of z and of t.
7. By simplifying the integral of Question 5 via the substitution of Question 6 derive the formula

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

8. By making the substitution $u = \frac{t}{1-t}$ show that

$$B(x,y) = \int_0^\infty \frac{t^{x-1}}{(1+t)^{x+y}}\,dx.$$

9. Show that $B(x,y) = B(x,y+1) + B(x+1,y)$.
10. And another identity:

$$B(x+1,y) = B(x,y)\cdot\frac{x}{x+y}.$$

Problem Set 7 Special Functions II: Chebyshev Polynomials

The form of the nth *Chebyshev polynomial of the first kind* $T_n(x)$ is defined by putting $x = \cos\theta$ and declaring that $T_n(\cos\theta) = \cos n\theta$ $(n \geq 0)$.

1. Find $T_0(x)$ and $T_1(x)$.
2. Find $T_2(x)$ and $T_3(x)$.
3. Show that in general $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$.
4. Use the result of Q3 to find $T_4(x)$.
5. Show that $T_n(T_m(x)) = T_{nm}(x)$.

Define the *Chebyshev polynomial of the second kind* $U_n(x)$ by

$$nU_{n-1}(x) = \frac{d(T_n(x))}{dx}, \ (n \geq 1).$$

6. Find $U_1(x)$, $U_2(x)$, and $U_3(x)$.
7. Show that
$$U_{n-1}(\cos\theta) = \frac{\sin n\theta}{\sin\theta}.$$
8. Show that
$$T_n(x) = U_n(x) - xU_{n-1}(x).$$
9. Show that
$$T_n(x) = \frac{1}{2}\big(U_n(x) - U_{n-2}(x)\big)$$
10. Show that $T_n(x)$ is a solution to the differential equation
$$(1 - x^2)y'' - xy' + n^2y = 0.$$

Problem Set 8 The Laplace transform

The *Laplace transform* of a function $f(t)$ is defined as

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\, dt.$$

.

1. Find $\mathcal{L}\{1\}$.
2. Show by induction that

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}},$$

for any non-negative integer n.

3. Find $\mathcal{L}\{e^{at}\}$ $(s > a)$.
4. Find $\mathcal{L}\{\sin at\}$.
5. Express $\mathcal{L}\{af(t) + bg(t)\}$ in terms of $\mathcal{L}\{f(t)\}$ and $\mathcal{L}\{g(t)\}$.
6. Express $\mathcal{L}\{f'(t)\}$ in terms of $\mathcal{L}\{(f(t)\}$.
7. Express $\mathcal{L}\{f''(t)\}$ in terms of $\mathcal{L}\{f(t)\}$.
8. Consider the equation $y'' - y' - 2y = 0$ with initial conditions $y(0) = 1$ and $y'(0) = 0$ and let $Y(s)$ be $\mathcal{L}\{y(t)\}$. Show that

$$Y(s) = \frac{s-1}{s^2 - s - 2}.$$

9. Use partial fractions to express $Y(s)$ in the form $\frac{a}{s-b} + \frac{c}{s-d}$.
10. Use the result of Question 9 to find the solution to the equation of Question 8.

Problem Set 9 Fourier series

Recall that the *Fourier series* of a function $f(x)$: if $f(x)$ is smooth function then $f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx)$ on the interval $[-\pi, \pi]$ where the Fourier coefficients are given by:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx,\; b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx$$

1. Calculate the Fourier series for the function $f(x) = x^2$.
2. Use the result of Question 1 to find the sum of the series

$$1 - \frac{1}{2^2} + \frac{1}{3^2} - \cdots .$$

3. Again using Question 1, find $\sum_{n=1}^{\infty} \frac{1}{n^2}$.
4. Find the Fourier series for $f(x) = |x|$ and hence find the sum of the series

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots .$$

5. Find the Fourier series for $f(x) = \cos \mu x$ for $-\pi < x < \pi$, where μ is *not* an integer.

6. Use Question 5 to find the *resolution of the cotangent into partial fractions*:

$$\cot \pi x - \frac{1}{\pi x} = -\frac{2x}{\pi}\left(\frac{1}{1^2 - x^2} + \frac{1}{2^2 - x^2} + \frac{1}{3^2 - x^2} + \cdots\right).$$

7. Prove by induction the *Lagrange identity*:

$$\frac{1}{2} + \cos u + \cos 2u + \cdots + \cos nu = \frac{\sin(n + \frac{1}{2})u}{2\sin\frac{1}{2}u}\;(u \neq n\pi).$$

8. Obtain the identity of Question 7 by summing the geometric series with $a = 1$ and $r = e^{iut}$, multiplying top and bottom by $e^{-\frac{1}{2}iu}$ and then taking the real part.

9. *Inversion of a Fourier series*: we shall find the function whose Fourier series is $f(t) = 1 + a\cos t + a^2\cos 2t + \cdots$ $(-1 < a < 1)$ in two steps as follows. First introduce the companion series $g(t) = a\sin t + a^2 \sin 2t + \cdots$ and show that $f(t) + ig(t) = \frac{1}{1-ae^{it}}$.

10. By passing to real and imaginary parts in the answer to Question 9, show that

$$f(t) = \frac{1 - a\cos t}{1 - 2a\cos t + a^2}.$$

Problem Set 10 Line integrals

For a real-vlaued function $f(x, y)$ and a curve C in $\mathbb{R}^2$parametrized by $x = x(t), y = y(t)$ for $a \leq t \leq b$ the *line integral* of $f(x, y)$ along C is

$$\int_C f(x,y)\,ds = \int_{t=a}^{b} f(x(t), y(t))\sqrt{\dot{x}^2(t) + \dot{y}^2(t)}\,dt.$$

For Questions 1 and 3, evaluate the given line integrals.

1.

$$\int_C x\,ds$$

where C is the curve $y = x^2$ for $-1 \leq x \leq 1$.

2. Take $f(x, y) = xy$ and C is the unit circle in the first quadrant, traced anti-clockwise.

3. $F(x, y) = x + y^2$, C is the curve that begins at $(2, 0)$, moves anti-clockwise around the circle $x^2 + y^2 = 4$ to the point $(-2, 0)$, and then returns to $(2, 0)$ along the x-axis.

4. Extend the notion of line integral to curves in three dimensions and evaluate

$$\int_C xyz\,ds$$

where C is the *helix* (spring shape) parametrized as $x(t) = \cos t$, $y(t) = \sin t$, $z(t) = 3t$ for $0 \leq t \leq 4\pi$.

For a *vector field* $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ and curve C with smooth parametrization $x = x(t), y = y(t)$ for $a \leq t \leq b$ the *line integral* I of $\mathbf{F}$ along C is I which equals

$$\int_C \mathbf{F}\bullet d\mathbf{r} = \int_C P(x,y)\,dx + \int_C Q(x,y)\,dy = \int_a^b ((P(x(t), y(t))\frac{dx}{dt} + Q(x(t), y(t)\frac{dy}{dt})\,dt$$

with a natural extension to curves in three dimensions. If $\mathbf{F}$ represents a force then the value of I is the *word done* by the force in moving along the curve in the given direction.

5. Show that an alternative formulation of the previous line integral is

$$\int_C \mathbf{F}\bullet d\mathbf{r} = \int_C \mathbf{F}\bullet \mathbf{T}ds$$

where $\mathbf{T}$ is the unit vector in the direction of the tangent to C at $(x(t), y(t))$.

In Questions 6-8 calculate the given line integrals for each of the vector fields $\mathbf{F}$.

6.

$$\mathbf{F}(x,y,z) = 8x^2yz\mathbf{i} + 5z\mathbf{j} - 4xy\mathbf{k}$$

and C is the curve parametrized by $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$ for $0 \le t \le 1$.

7.

$$\mathbf{F}(x,y,z) = xz\mathbf{i} - yz\mathbf{k}$$

and C is the line segment from $(-1,2,0)$ to $(3,0,1)$.

8. *Integrals independent of parametrization* Suppose that for our line integal as above we have $t = \alpha(u)$ with $\alpha'(u) > 0$ with $a = \alpha(c), b = \alpha(d)$ say and write $x = \tilde{x}(u) = x(\alpha(t)), y = \tilde{y}(u) = y(\alpha(t))$. Denote $\frac{dt}{du} = \frac{d\alpha}{du}$ by $\alpha'(u)$. Show that the value of $\int_C P(x,y)\,dx$ has the same value irrespective of whether the parameter t or u is used to calculate it.

9. Repeat Question 8 for the integral of a real-valued function $\int_C f(x,y)\,ds$.

10. Suppose that C is the curve from $(0,14)$ to (1.8) with parametrization $\mathbf{r}(t) = (t+3, t^2-t+2)$ for $-3 \le t \le -2$. Let $\mathbf{F}(x,y) = (2x^2, -y)$. Show that the parametrization $\mathbf{r}(u) = (u, u^2 - 7u + 14)$, $0 \le u \le 1$ describes the same curve C and calculate $\int_C \mathbf{F}\bullet d\mathbf{r}$ using both parametrizations, verifying the general result shown in Question 8 in this particular case.

Hints for Problems

Problem Set 1

8. The upper limit of z for an arbitrary point (x, y) in the base of the volume in the xy-plane is $1 - x - y$. For the upper limit of the second integral, take the projection of the plane into the xy-plane and we need the greatest value of y for an arbitrary value of x.

10. The integral that arises involves transforming to polar variables with $dxdy$ being replaced by $rdrd\theta$ (see Set 2 for details).

Problem Set 2

2. Sketch the curve and find the area of one leaf, taking care to find the value of the upper limit of θ.

3 & 4. Work in polars.

4. Write the integral of Question 3 as a product of two integrals, identical except for the name of the variable of integration.

6. Again, work in polars, so you will need the equation of the boundary semicircle in polar form in order to find the limits of integration.

8. Integrate $r\,dzdrd\theta$, writing the upper limit for z in terms of r.

Problem Set 3

1. Differentiate $\tan(\phi(t))$ by the Chain rule and then write $\dot{\phi}$ in terms of $\dot{x}$ and $\dot{y}$.

2. This time use $\frac{d\phi}{ds} = \frac{d\phi}{dt} \cdot \frac{dt}{ds}$ and write eveything in terms of $\dot{x}$ and $\dot{y}$.

8. The required integral $\int y\,dx$ over one arch is, in terms of the parameter $\int_0^{2\pi} y(t)\frac{dx}{dt}\,dt$.

9. Apply the Fundamental theorem of calculus and the formulae for arc length.

Problem Set 4

3. Apply L'Hopital's rule twice to the limit that arises in order to find a suitable l.

4 - 7. These are 'if and only if' statements so require separate arguments, working from the definitions, in each direction. The rule of Question 6 is partiularly useful, often in the special form $fo(g) = o(fg)$.

10. A typical application: expand the product and use the rules of the previous questions, trying to absorb smaller terms into larger ones; for example, as $n \to \infty$, $O(n^3) + O(n^2) = O(n^3)$.

Problem Set 5

5. Find $g^{(n)}(x)$ directly by integrating first and then by differentiating through the integral and equate the two forms.

9. Look to the derivative of $\cos x - (1 - \frac{x^2}{2})$.

10. Make use of Question 9 and integrate between 1 and $\varepsilon > 0$.

Problem Set 6

6. $0 \leq z \leq \infty$, $0 \leq t \leq 1$.

Problem Set 7

3. Make use of $\cos A + \cos B = 2 \cos \frac{A+B}{2} \cos \frac{A-B}{2}$.

7. Chain Rule.

8. & 9. Make use of Question 7.

Problem Set 8

2. Integrate by parts.

10. Take the inverse Laplace transform and the result of Question 5.

Problem Set 9

Remember that an even (resp. odd) function has only a cosine (resp. sine) series for its Fourier expansion.

2,3, 5, and 6. Choose special values for x to derive the particular series results required.

8. Multiply top and bottom by $e^{-\frac{1}{2}iu}$ and then take the real part.

10. Carry out the division in the usual way by multiplying top and bottom by the complex conjugate and then take the real part.

Problem Set 10

3. Extend the definition to a curve in 3-space.

4. Calculate the integrals over the two separate curves and sum to get the answer.

6. $\int_C \mathbf{F}\bullet d\mathbf{r} = \int_a^b \mathbf{F}(r(t))\bullet\mathbf{r}'(t)\,dt$

8 & 9. In each case re-write the line integral in terms of the give substitution using the Chain rule.

10. Identitfy the function $t = \alpha(u)$ by solving $x(t) = x(\alpha(u)) = \tilde{x}(u)$, checking that $\frac{d\alpha}{du} > 0$, and that $y(t) = y(\alpha(u)) = \tilde{y}(u)$ also holds. Find the limits of integration when using the parameter u, checking that they indeed correspond to the endpoints of C taken in the correct order.

Answers to the Problems

Problem Set 1

1. $\frac{5}{3}$. 2. $\frac{\sqrt{2}}{8}(4-\pi)$. 3. $\frac{e-1}{3}$. 4. 1. 5. $e-1$. 6. $2(e-2)$. 7. $\frac{1}{3}$. 8. $\frac{1}{6}$. 9. $\frac{4\pi}{3}$. 10. $\frac{52\pi}{3}$.

Problem Set 2

1. r. 2. $\frac{\pi}{4}$. 3. π. 4. $\sqrt{\pi}$. 5. $\sqrt{2\pi}$. 6. $\frac{\pi+2}{2}$. 7. $\frac{3\pi}{2}$. 8. 8π. 9. $r^2 \sin\phi$. 10. $\frac{\pi}{3}\rho_0 a^3$.

Problem Set 3

3. $\frac{1}{a}$. 4. $\frac{2}{(1+4x^2)^{\frac{3}{2}}}$. 5. $-\cos x$. 6. $8a$. 7. $-4a|\sin\frac{t}{2}|$. 8. $3a^2\pi$. 9. t_0. 10. 2.

Problem Set 4

10. $n^3 + O(n^{\frac{5}{2}})$.

Problem Set 5

3. $\log(b+1)$. 4. $\log 3$. 6. $f(b) = C - \arctan b$. 7. $C = \frac{\pi}{2}$. 8. π. 10. No, the (improper) integral diverges.

Problem Set 6

As per problem set.

Problem Set 7

1. $1, x$. 2. $2x^2-1,\ 4x^3-3x$. 4. $8x^4-8x^2+1$. 6. $2x,\ 4x^2-1,\ 8x^3-2$.

Problem Set 8

1. $\frac{1}{s}$. 2. $\frac{n!}{s^{n+1}}$. 3. $\frac{1}{s-a}$. 4. $\frac{a}{a^2+s^2}$. 5. $a\mathcal{L}\{f(t)\}+b\mathcal{L}\{g(t)\}$. 6. $\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\}-f(0)$. 7. $\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\}-sf(0)-f'(0)$. 8. $\frac{s-1}{s^2-s-2}$. 9. $\frac{1/3}{s-2}+\frac{2/3}{s+1}$. 10. $y(t)=\frac{1}{3}e^{2t}+\frac{2}{3}e^{-t}$.

Problem Set 9

1. $x^2=\frac{\pi^2}{3}+\sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2}\cos nx$. 2. $\frac{\pi^2}{12}$. 3. $\frac{\pi^2}{6}$. 4. $\frac{\pi}{2}-\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2}, \frac{\pi^2}{8}$. 5. $\frac{2\mu\sin\mu\pi}{\pi}\left(\frac{1}{2\mu^2}-\frac{\cos x}{\mu^2-1^2}+\frac{\cos 2x}{\mu^2-2^2}-\cdots\right)$. 9. $\frac{1}{1-ae^{it}}$. 10. $f(t)=\frac{a\sin t}{1-2a\cos t+a^2}$.

Problem Set 10

1. 0. 2. $\frac{1}{2}$. 3. 4π. 4. $-3\sqrt{10}\pi$. 6. 1 . 7. 3. 10. $66\frac{2}{3}$.