Mathematics 107 Further Calculus & Differential equations

Professor Peter M. Higgins

March 9, 2018

Solutions and Comments

Problem Set 1 Approximations

1.

$$f(x) = x^{-1} \Rightarrow f'(x) = -\frac{1}{x^2}; \text{ and } f'(\tfrac{1}{2}) = -2^2 = -4.$$

Also $f(\frac{1}{2}) = 2$. Hence

$$L(x) = 2 + (x - \frac{1}{2})(-4) = -4x + 4, \text{ or } L(x) = 4(1 - x).$$

2. $V = \frac{4}{3}\pi r^3 \Rightarrow V' = \frac{dV}{dr} = 4\pi r^2$. Hence $V'(5) = 4\pi(5^2) = 100\pi$ and we obtain the decrease in volume $V(5) - V(4 \cdot 92)$ is approximately equal to:

$$V'(5) \times 0 \cdot 08 = 8\pi = 25 \cdot 13\,\text{cm}^3.$$

3. $f(x) = x^{\frac{1}{2}} \Rightarrow f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$. Hence $f'(25) = \frac{1}{2\sqrt{25}} = \frac{1}{10}$. Therefore

$$\sqrt{26} \approx \sqrt{25} + \frac{1}{10}(26 - 25) = 5 + 0 \cdot 1 = 5 \cdot 1.$$

(Note that $(5 \cdot 1)^2 = 26 \cdot 01$).

4. Here we have $h = \frac{b-a}{n} = \frac{2-1}{8} = \frac{1}{8}$. Also $x_i = 1 + \frac{i}{8} = \frac{8+i}{8}$. Hence $y_i = f(x_i) = \frac{1}{x_i} = \frac{8}{8+i}$ $(1 \le i \le 8)$. Hence the value of S_8 is given by:

$$\frac{1}{24}\Big[1 + 4\Big(\frac{8}{8+1} + \frac{8}{8+3} + \frac{8}{8+5} + \frac{8}{8+7}\Big) + 2\Big(\frac{8}{8+2} + \frac{8}{8+4} + \frac{8}{8+6}\Big) + \frac{8}{8+8}\Big]$$

$$= \frac{1}{24}\Big(1 + 4\Big(\frac{8}{9} + \frac{8}{11} + \frac{8}{13} + \frac{8}{15}\Big) + \Big(\frac{8}{5} + \frac{8}{6} + \frac{8}{7}\Big) + \frac{1}{2}\Big) =$$

$$= 0.693154...$$

This compares with the exact value $\ln 2 = 0 \cdot 693147....$

5. We have $f'(x) = -x^{-2}$, $f''(x) = 2x^{-3}$, $f^{(3)}(x) = -6x^{-4}$, $f^{(4)}(x) = 24x^{-5}$. Hence on $[1, 2]$ we have $|f^{(4)}(x)| \le \frac{24}{1^5} = 24$.

6. The error term in S_8 is therefore bounded by $\frac{24\times(2-1)^5}{180\times 8^4} = \frac{1}{15\times 4\times 8^3} = \frac{1}{30{,}720} = 0 \cdot 0000326$ (3 sf).

Comments Named after Thomas Simpson (1710-61), the rule is a staple of mathematical engineering. It is however a form of the three-point Newton-Cotes quadrature rule and similar rules were used by Kepler a century before Simpson. Simpson's rule works by approximating the curve by a parabola that matches the curve at the endpoints and the midpoint of each interval. A simpler rule, the *Trapezium Rule*, approximates the curve by the straight line between endpoints but this is generally less accurate, although the Trapezium rule can give excellent results for periodic functions across a period.

7. The exact answer in this case is $[x^5]_0^1 = 1$. The Simpson's estimate is:

$$S_4 = \frac{1}{12}(0 + 4(\frac{5}{256}) + 2(\frac{80}{256}) + 4(\frac{405}{256}) + 5) = 1 \cdot 00260;$$

hence the exact error is 0·00260. Now $f^{(4)}(x) = 120$ we take $K = 120$, $(b-a) = 1$ and $h = \frac{1}{4}$ to get as our error bound:

$$|E_S| \leq \frac{120(1)}{180}(\frac{1}{4})^4 = \frac{1}{384} < 0 \cdot 00261.$$

8. First observe that $f(1) = 1-1-1 = -2 < 0$ while $f(2) = 8-2-1 = 5 > 0$ so that there exists at least one root r in the interval $(1, 2)$. We also have $f'(x) = 3x^2 - 1$, so the Newton-Raphson formula becomes

$$x_{n+1} = x_n - \frac{x_n^3 - x_n - 1}{3x_n^2 - 1} = \frac{2x_n^3 + 1}{3x_n^2 - 1}.$$

Putting $x_0 = 1 \cdot 5$ we get (just recording to 4d.p.) as the successive values of x_n : $x_1 = 1 \cdot 3478$, $x_2 = 1 \cdot 3252$, $x_3 = 1 \cdot 3247$, $x_4 = 1 \cdot 3247$, $x_5 = 1 \cdot 32471795724 \cdots$. We find that $f(x_5)$ equal 0 to 10 decimal places, so $r = 1 \cdot 32471795724$ rounded to 10 decimal places.

9. We use $f(x) = x^3 - \cos x$. We get from the Newton-Raphson formula:

$$x_{n+1} = x_n - \frac{x_n^3 - \cos x_n}{3x_n^2 + \sin x_n} = \frac{2x_n^3 + x_n \sin x_n + \cos x_n}{3x_n^2 + \sin x_n}.$$

Starting with $x_0 = 0 \cdot 8$, the successive approximations (to 4 d.p.) are $x_1 = 0 \cdot 8700$, $x_2 = 0 \cdot 8655 = x_3 = x_4$, $x_5 = 0 \cdot 865474033102 \cdots$. We find $f(x_5) = 0 \cdot 00 \cdots$ equals zero to 12 decimal places, so $r = 0 \cdot 865474033102 \cdots$ is our approximate root satisfying $r^3 = \cos r$.

Comment The idea behind the method is to find better and better approximations to a nearby root by calculating and re-calculating the x-intercept of the tangent to the curve at $(x_i, f(x_i))$: although convergence is not guaranteed, typically the number of correct digits in the approximation increases by 2 on each iteration.

10. To four decimal places, the outcome of hitting the cosine button again and again is the number $c = 0 \cdot 7391 (\approx 42.37°)$. When the display stabilizes at c, we must therefore have $c = \cos c$ so that $x = c$ is the (unique) solution of the equation $x = \cos x$.

Comment: The initial value t that you choose hardly matters as, no matter which number t you select, $-1 \leq \cos t \leq 1$; since $\cos 0 = 1$ and $\cos x = \cos(-x)$ is always true, in effect, after one iteration you have a new number, t', in the range $\cos 1 = 0 \cdot 5403 \leq t' \leq 1$.

Finding a fixed point in this fashion will work with any function $f(x)$ that satisfies the inequality:

$$|f(x) - f(y)| < |x - y|.$$

The fact that the cosine function behaves this way stems from the property that for $x, y \in [-1, 1]$ we have $|\cos x - \cos y| < |x - y|$, a fact that can be deduced from the identity:

$$\cos x - \cos y = -2\sin(\frac{x+y}{2})\sin(\frac{x-y}{2}).$$

Problem Set 2 Integration

1.
$$\frac{1}{1-x^2} = \frac{1}{(1-x)(1+x)} = \frac{1}{2(1+x)} + \frac{1}{2(1-x)}, \text{ Hence}$$

$$\int \frac{dx}{1-x^2} = \frac{1}{2}\int \frac{dx}{1+x} + \frac{1}{2}\int \frac{dx}{1-x} = \frac{1}{2}\big(\ln|1+x| - \ln|1-x|\big) + c = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| + c.$$

2. Put $x = \frac{1}{2}\sinh u$ so that $dx = \frac{1}{2}\cosh u\, du$ and $\sqrt{1+4x^2} = \sqrt{1+\sinh^2 u} = \sqrt{\cosh^2 u} = \cosh u$ as $\cosh u > 0$. When $x = 0$ and $x = 1$ we have respective values of u satisfying $0 = \sinh u \Rightarrow u = 0$ and $1 = \frac{1}{2}\sinh u \Rightarrow u = \sinh^{-1}(2)$. Therefore our integral becomes:

$$\int_0^{\sinh^{-1}(2)} \frac{2}{\cosh u} \cdot \frac{1}{2}\cosh u\, du = \int_0^{\sinh^{-1}(2)} du = \sinh^{-1}(2).$$

3. We have that $\sinh x < \cosh x$ so that the increment of volume by cylindrical shells is given by $(\pi x^2)(\cosh x - \sinh x)$. Now $\cosh x - \sinh x = \frac{e^x+e^{-x}}{2} - \frac{e^x-e^{-x}}{2} = e^{-x}$. Hence we require

$$\pi\int_0^{\infty} x^2(\cosh x - \sinh x)\, dx = \pi\int_0^{\infty} x^2 e^{-x}\, dx.$$

Drop the factor π for the moment and integrate our integral I by parts with $u = x^2$, $dv = e^{-x}\, dx \Rightarrow du = 2xdx$ and $v = -e^{-x}$ so that $I = -x^2e^{-x} + \int 2xe^{-x}\, dx$. Integrating again by parts in a similar fashion we obtain:

$$I = -x^2e^{-x} + \big(-2xe^{-x} + 2\int e^{-x}\, dx\big) = -x^2e^{-x} - 2xe^{-x} - 2e^{-x}.$$

Evaluating between 0 and ∞ now gives the area A as:

$$A = -\pi\big[e^{-x}(x^2 + 2x + 2)\big]_0^{\infty} = -\pi\big[0 - (1(0 + 0 + 2))\big] = 2\pi.$$

4. One line of division immediately gives $y = 2x + \frac{5x-3}{x^2-2x-3}$. The denominator factorizes as $x^2 - 2x - 3 = (x-3)(x+1)$. Hence the graph of $y(x)$ has vertical asymptotes at $x = -1$ and $x = 3$ and an oblique asymtptote in the line $y = 2x$ which is approached for large values of $|x|$.

5. $\frac{5x-3}{(x-3)(x+1)} \equiv \frac{A}{x-3} + \frac{B}{x+1}$. By the Cover-up method we get $A = \frac{5(3)-3}{3+1} = \frac{12}{4} = 3$ while $B = \frac{5(-1)-3}{-1-3} = \frac{8}{4} = 2$: hence

$$\frac{5x-3}{x^2-2x-3} = \frac{3}{x-3} + \frac{2}{x+1}.$$

6. From Question 4 and 5 we obtain:

$$\int \frac{2x^3-4x^2-x-3}{x^2-2x-3}\,dx = \int (2x + \frac{3}{x-3} + \frac{2}{x+1})\,dx =$$

$$x^2 + 3\ln|x-3| + 2\ln|x+1| + c.$$

7. By a standard identity we have $\sqrt{\frac{1+\cos 2x}{2}} = \sqrt{\cos^2 x} = |\cos x|$. Hence our integral becomes

$$\int_0^{\pi} |\cos x|\,dx = 2\int_0^{\frac{\pi}{2}} \cos x\,dx = 2[\sin x]_0^{\frac{\pi}{2}} = 2[1-0] = 2.$$

8. Put $u = 2 - \cos x \Rightarrow du = \sin x\,dx$; when $x = 0$, $u = 2 - 1 = 1$, and when $x = \pi$, $u = 2 - \cos\pi = 2 - (-1) = 3$. Hence the integral becomes

$$\int_1^3 \frac{du}{u} = \ln|u|\Big|_1^3 = \ln 3 - \ln 1 = \ln 3.$$

9. Integrate by parts by letting $u = x^{m-1}$,

$$dv = e^{-x} \Rightarrow du = (m-1)x^{m-2}\,dx, v = -e^{-x}.\text{We obtain:}$$

$$F(m) = -[x^{m-1}e^{-x}]_0^{\infty} + \int_0^{\infty} (m-1)x^{m-2}e^{-x}\,dx = (m-1)F(m-1).$$

$$\text{Hence}\ \frac{F(m)}{F(m-1)} = m-1.$$

Comment The function $F(m)$ defined above is known as the *beta function,* a special function that arises throughout mathematics and physics. The beta function allows for a continuous generalization of the notion of factorial to a function on the real line.

10. Volume is given by

$$\int_0^4 \pi y^2\,dx = \pi\int_0^4 x\,dx = \pi\big[\frac{x^2}{2}\big]_0^4 = \pi\big[\frac{16}{2} - 0\big] = 8\pi.$$

Problem Set 3 Limits

1.

$$\lim_{x\to\infty}\frac{3x^3-18x-1}{-6x^3+x^2}=\lim_{x\to\infty}\frac{3-\frac{18}{x^2}-\frac{1}{x^3}}{-6+\frac{1}{x}}=\frac{3}{-6}=-\frac{1}{2}.$$

2.

$$\lim_{x\to 0}\frac{\sin 7x}{x}=7\lim_{x\to 0}\frac{\sin 7x}{7x}=7\lim_{y\to 0}\frac{\sin y}{y}\text{ where } y=7x$$

$$=7\cdot 1=7.$$

3.

$$\lim_{x\to 0}\frac{1-\cos x}{x}=\lim_{x\to 0}\frac{2\sin^2(\frac{x}{2})}{x}=(\lim_{x\to 0}\sin x)(\lim_{x\to 0}\frac{\sin(x/2)}{x/2})=0\cdot 1=0.$$

Comment: the limits of Questions 2 & 3 are required in order to show from first principles that $(\sin x)'=\cos x$, from which derivatives of all trigonometric functions can be got. The evaluation of $\sin x/x$ as $x\to 0$ is found by a sandwich argument based on associated areas of triangles and corresponding sectors of the unit circle. Since the expression in Question 3 is a $\frac{0}{0}$ form, L'Hopital's Rule can also be used: differentiating top and bottom gives an equal limit, which is in this case: $\lim_{x\to 0}\frac{\sin x}{1}=0$. However, this pre-supposes we have the derivatives of trigonometric functions to hand.

4.

$$\lim_{n\to\infty}(n(1+\frac{1}{n})-n)=\lim_{n\to\infty}(n+1-n)=\lim_{n\to\infty}(1)=1.$$

Comment Must avoid the sloppy argument that $1+\frac{1}{n}\to 1$ so the limit is $n-n=0$.

5.

$$\lim_{x\to 0}x\ln x=\lim_{x\to 0}\frac{\ln x}{x^{-1}}=\lim_{x\to 0}\frac{(\ln x)'}{(x^{-1})'}$$

$$=-\lim_{x\to 0}\frac{x^{-1}}{x^{-2}}=-\lim_{x\to 0}x=-0=0.$$

Comment: here we are using L'Hopital's rule concerning limits of indeterminant ratios and products. We also make use of elementary properties of limits, those being that the operation taking limits commutes with arithmetic operations and with continuous functions.

6.

$$y=\lim_{x\to 0}x^x\Rightarrow\ln y=\ln(\lim_{x\to 0}x^x)=\lim_{x\to 0}(\ln x^x)$$

$$=\lim_{x\to 0}(x\ln x)=0.\text{ Hence } y=e^0=1.$$

7.

$$y = \lim_{n\to\infty} n^{\frac{1}{n}} \Rightarrow \ln y = \lim_{n\to\infty} (\ln n^{\frac{1}{n}}) = \lim_{n\to} \frac{\ln n}{n} = 0$$
$$\Rightarrow y = e^0 = 1.$$

8.

$$\lim_{n\to\infty} (1 + \frac{1}{2n})^n = (\lim_{n\to\infty} (1 + \frac{1}{2n})^{2n})^{\frac{1}{2}} = \sqrt{e}.$$

Comment: we are making use of the standard limit of $e^x = \lim_{n\to\infty}(1+\frac{x}{n})^n$.

9.

$$\lim_{n\to\infty} (\sqrt{n+1} - \sqrt{n}) = \lim_{n\to\infty} \frac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+1} + \sqrt{n}}$$
$$= \lim_{n\to\infty} \frac{n+1-n}{\sqrt{n+1} + \sqrt{n}} = \lim_{n\to\infty} \frac{1}{\sqrt{n+1} + \sqrt{n}} = 0.$$

.

10.

$$\lim_{x\to 0^+} \frac{\sin\sqrt{x}}{x} = \lim_{x\to 0} \frac{\cos x^2}{2\sqrt{x}\cdot 1} = +\infty.$$

Problem Set 4 Differentiation

1. $y = \cosh^{-1} x$ $(x \geq 1)$so write

$$x = \cosh y \Rightarrow \frac{dx}{dy} = \sinh y \Rightarrow \frac{dy}{dx} = \frac{1}{\sinh y}$$
$$= \frac{1}{\sqrt{\cosh^2 y - 1}} = \frac{1}{\sqrt{x^2-1}}\ (x > 1).$$

Comment At $x = 1$ we have $\cosh^{-1}(1) = 0$ and the tangent line to the curve $y = \cosh^{-1} x$ is vertical, so no derivative exists.

2. We are given that $\frac{dr}{dt} = 0 \cdot 1$ and $V = \frac{4}{3}\pi r^3$, where r and V are the respective radius and volume of the bubble. Hence $\frac{dV}{dr} = 4\pi r^2$. By the Chain Rule we get

$$\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt} = (4\pi r^2)(0 \cdot 1) = 0 \cdot 4\pi r^2.$$

We require

$$\frac{dV}{dt}\Big|_{r=0\cdot 8} = 0 \cdot 4\pi(0 \cdot 8)^2 = 0 \cdot 256\pi \approx 0 \cdot 8042 \text{ cm}^3/\text{sec}.$$

3. Writing s for the separation of the lorries we have the equation $s^2 = x^2 + y^2$ where the x and y directions correspond to east and north. Differentiating with respect to time and dividing through by 2 yields:

$$s\dot{s} = x\dot{x} + y\dot{y} \Rightarrow \dot{s} = \frac{x\dot{x} + y\dot{y}}{s}.$$

Now 6 minutes corresponds to $\frac{6}{60} = 0 \cdot 1$ hours. We have $x(0 \cdot 1) = 30 \times 0 \cdot 1 = 3$, $y(0 \cdot 1) = 40 \times 0 \cdot 1 = 4$. Hence after 6 minutes we have $s^2 = x^2 + y^2 = 9 + 16 = 25$, so that $s = 5$. Moreover $\dot{x} = 30$ and $\dot{y} = 40$ are given. We may now evaluate:

$$\dot{s}|_{t=0\cdot 1} = \frac{3(30) + 4(40)}{5} = \frac{90 + 160}{5} = \frac{250}{5} = 50\text{mph}.$$

Comment Alternatively, using a velocity vector diagram, we can see that we have essentially a $3, 4, 5$ triangle of velocities.

4. We write $A = xy$ so that $\dot{A} = x\dot{y} + \dot{x}y$. We want to evaluate $\dot{A}$ when $x = 15, \dot{x} = 3$and $y = 6, \dot{y} = 2$, so that $\dot{A} = (15)(2) + (3)(6) = 30 + 18 = 48\text{m}^2/\text{sec}$.

5. For $y = x^3$ we have $y' = 3x^2$ while for $y = \sqrt{x}$ we have $y' = \frac{1}{2}x^{-\frac{1}{2}}$. At the common point $(1, 1)$ we have $\tan\theta_1 = 3(1^2) = 3$ and $\tan\theta_2 = \frac{1}{2}(1)^{-\frac{1}{2}} = \frac{1}{2}$, where θ_1, θ_2 are the respective angles that the tangents make with the x-axis. Our required angle is then $\theta_1 - \theta_2$ and we have by a standard identity:

$$\tan(\theta_1 - \theta_2) = \frac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1 \tan\theta_2} = \frac{3 - \frac{1}{2}}{1 + \frac{3}{2}} = \frac{6 - 1}{2 + 3} = \frac{5}{5} = 1,$$

and so the angle between the two tangents is $45°$.

6.
$$\frac{dz}{dt} = \frac{1}{2}(xy + y)^{-\frac{1}{2}}(y)(-\sin t) + \frac{1}{2}(xy + y)^{-\frac{1}{2}}(x + 1)\cos t$$

when $t = \frac{\pi}{2}$,we have $x = \cos\frac{\pi}{2} = 0$, and $y = \sin\frac{\pi}{2} = 1$. Substituting $x = 0$,$y = 1$ and $t = \frac{\pi}{2}$ gives us

$$\frac{dz}{dt}|_{t=\frac{\pi}{2}} = \frac{1}{2}(1)(1)(-1) + \frac{1}{2}(1)(1)(0) = -\frac{1}{2}.$$

7. $\frac{1}{u} + \frac{1}{v} = \frac{1}{f} = 6$, $\dot{u}(t) = 4$. Differentiating the equation with respect to time gives

$$\frac{1}{u^2}\frac{du}{dt} + \frac{1}{v^2}\frac{dv}{dt} = 0 \Rightarrow \frac{dv}{dt} = -\frac{v^2}{u^2}\frac{du}{dt}.$$

Now $u = 5 \Rightarrow \frac{1}{v} = \frac{1}{6} - \frac{1}{5} = \frac{1}{30} \Rightarrow v = 30$. We require

$$\frac{dv}{dt}|_{u=5} = -\frac{30^2}{5^2} \cdot 4 = -144\text{cm}/\text{sec}.$$

That is to say the image is retreating at a rate of 144cm/sec.

8. Putting θ equal to the angle between the ground and the ladder, the length of the ladder is given by $L(\theta) = \frac{1}{\cos\theta} + \frac{2}{\sin\theta}$. Since $0 < \theta < \frac{\pi}{2}$ and since $L(\theta) \to \infty$ as θ approaches either 0 or $\frac{\pi}{2}$, it follows that there is a minimum value that can be found by solving $L'(\theta) = 0$:

$$0 = L'(\theta) = \frac{\sin\theta}{\cos^2\theta} - \frac{2\cos\theta}{\sin^2\theta} = \frac{\sin^3\theta - 2\cos^3\theta}{\cos^2\theta\sin^2\theta}.$$

At any critical points we have $\sin^3\theta = 2\cos^3\theta \Leftrightarrow \tan^3\theta = 2$. Now $\sec^2\theta = 1 + \tan^2\theta = 1 + 2^{\frac{2}{3}}$ for the critical value of θ. Continuing we have:

$$\cos\theta = \frac{1}{(1+2^{\frac{2}{3}})^{\frac{1}{2}}} \text{ and } \sin\theta = \tan\theta\cos\theta = \frac{2^{\frac{1}{3}}}{(1+2^{\frac{2}{3}})^{\frac{3}{2}}}.$$

Therefore the minimum value of $L(\theta)$ is

$$\frac{1}{\cos\theta} + \frac{2}{\sin\theta} = (1+2^{\frac{2}{3}})^{\frac{1}{2}} + 2\frac{(1+2^{\frac{2}{3}})^{\frac{1}{2}}}{2^{\frac{1}{3}}} = (1+2^{\frac{2}{3}})^{\frac{3}{2}} \approx 4\cdot 16\text{m}.$$

9. Let us take the shoreline for the x-axis with the origin at the point where a right-angle to the axis passes through the lighthouse. Let the increasing direction of x be the same direction as the beam moves along the shore and let ω be the angle that the ray from the beacon makes with the y-axis. Then $\tan\omega = \frac{x}{4} \Rightarrow x = 4\tan\omega$. Hence

$$\dot{x} = 4\sec^2\omega\cdot\dot{\omega}.$$

We need to evaluate $\dot{x}$ when $\omega = \frac{\pi}{4}$. Now $\dot{\omega}$ is the constant angular velocity of $\frac{2\pi}{10} = \frac{\pi}{5}$ radians/sec. Hence the required value of $\dot{x}$ is

$$4(\sqrt{2})^2\frac{2\pi}{10} = \frac{16\pi}{10} = \frac{8\pi}{5}\text{km/sec}.$$

10. $f(x) = e^x + x$ so

$$f^{-1}(x)'|_{f(x)} = \frac{1}{f''(x)}|_x = \frac{1}{e^x+1},$$

putting $f(x) = f(\ln 2)$ in this expression then gives

$$\frac{1}{e^{\ln 2}+1} = \frac{1}{2+1} = \frac{1}{3}.$$

Problem Set 5 Integration

1. Put $u = x^n$ so that $du = nx^{n-1}$ and $dv = e^x dx$ so that $v = e^x$. Integrating by parts then gives $I = x^n e^x - n\int x^{n-1}e^x\,dx$. Hence

$$I_n = x^n e^x - nI_{n-1}.$$

2. Using the result of Question 1 we obtain

$$I_3 = x^3e^x - 3I_2 = x^3e^x - 3(x^2e^x - 2I_1) = x^3e^x - 3x^2e^x + 6I_1 = x^3e^x - 3x^2e^x + 6(xe^x - I_0)$$

$$= x^3e^x - 3x^2e^x + 6xe^x - 6\int e^x\,dx. \text{ Therefore}$$

$$\int x^3e^x\,dx = e^x(x^3 - 3x^2 + 6x - 6) + c.$$

3. Integrate by parts by putting $dv = dx$ so that $v = x$ and $u = (\sin^{-1} x)^n$ so that $du = n(\sin^{-1} x)^{n-1}\frac{dx}{\sqrt{1-x^2}}$. Hence our integral I becomes:

$$I = x(\sin^{-1} x)^n - n\int (\sin^{-1} x)^{n-1}\frac{x\,dx}{\sqrt{1-x^2}};$$

for this new integral J, integrate by parts again with $u = (\sin^{-1} x)^{n-1}$ so that $du = (n-1)\frac{(\sin^{-1} x)^{n-2}}{\sqrt{1-x^2}}\,dx$ and $dv = \frac{x\,dx}{\sqrt{1-x^2}}$ so that $v = -\sqrt{1-x^2}$. Our integral J then becomes:

$$J = -\sqrt{1-x^2}(\sin^{-1} x)^{n-1} + (n-1)\int (\sin^{-1} x)^{n-2}\,dx; \text{ and hence}$$

$$I = x(\sin^{-1} x)^n + n\sqrt{1-x^2}(\sin^{-1} x)^{n-1} - n(n-1)\int (\sin^{-1} x)^{n-2}\,dx.$$

4. Putting $n = 2$ in the recursion of Question 3 now gives:

$$\int (\sin^{-1} x)^2\,dx = x(\sin^{-1} x)^2 + 2\sqrt{1-x^2}\sin^{-1} x - 2\int dx =$$

$$x(\sin^{-1} x)^2 + 2\sqrt{1-x^2} - 2x.$$

5. Applying the arc length formula for $y = x^2$ we get $L = \int_0^1 \sqrt{1+4x^2}dx$. Put $x = \frac{1}{2}\sinh t$ we get $dx = \frac{1}{2}\cosh t\,dt$,

$$\sqrt{1+4x^2} = \sqrt{1+\sinh^2 t} = \sqrt{\cosh^2 t} = \cosh t.$$

Using $t = \sinh^{-1}(2x)$ we get a lower limit of $t = \sinh^{-1}(0) = 0$ and an upper limit of $t = \sinh^{-1}(2)$. Hence we obtain:

$$L = \frac{1}{2}\int_0^{\sinh^{-1}(2)} \cosh^2 t\,dt = \frac{1}{2}\int_0^{\sinh^{-1}(2)} (1 + \cosh 2t)\,dt =$$

$$\frac{1}{2}[t + \frac{1}{2}\sinh 2t]_0^{\sinh^{-1}(2)} = \frac{1}{2}[\sinh^{-1}(2) + \cosh(\sinh^{-1}(2))\sinh(\sinh^{-1}(2)) - (0+0)].$$

Now $\cosh^2 t = 1 + \sinh^2 t$ so that $\cosh^2(\sinh^{-1}(2)) = 1 + 2^2 = 5$. Hence

$$L = \frac{1}{2}[\sinh^{-1}(2) + 2\sqrt{5}] = \frac{\sinh^{-1}(2) + \sqrt{5}}{2}.$$

6.

$$L = \int_0^{\pi/2} \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\,dt = \int_0^{\pi/1} \sqrt{a^2\sin^2 t + a^2\cos^2 t + b^2}dt$$

$$\int_0^{\pi/2} \sqrt{a^2+b^2}dt = \frac{\pi}{2}\sqrt{a^2+b^2}.$$

7. The integrand diverges to infinity at $x = 2$ so we must split the integral into: $I + J = \int_1^2 + \int_2^4$. Now

$$I = \lim_{l\to 2^-} \int_1^l \frac{dx}{(x-2)^{2/3}} = \lim_{l\to 2^-} [3(l-2)^{1/3} - 3(1-2)^{1/3}] = 3,$$

$$J = \lim_{l\to 2^+} [3(4-2)^{1/3} - 3(l-2)^{1/3}] = 3(2^{1/3}).$$

Thus the answer is $I + J = 3(1 + 2^{1/3})$.

Comment: When evaluating an integral $\int_a^b f(x)dx = F(b) - F(a)$ we are using the Fundamental Theorem of Calculus, whose proof requires that $f(x)$ be continuous along the interval (a, b). If this is not true, like in this example, it may not work. To see an instance of failure, take $\int_0^2 \frac{dx}{(x-1)^2}$. If we ignore the singularity at $x = 1$ and evaluate $-\frac{1}{x-1}\Big|_0^2 = -1-(1) = -2$, we have a nonsense answer as the integrand is always positive where it is defined so the integral cannot be negative. If you evaluate each part of this integral either side of the singularity you will see that each diverges to $+\infty$ as the troublesome point $x = 1$ is approached.

8. Substitute $y = -x$ into the first integral I so that $dx = -dy$; we get

$$I = -\int_a^{-a} f(y)dy = \int_{-a}^{a} f(y)dy$$

which, up to the naming of variables, is the integral on the right

9. Denote the integral by I and use the result of Question 8 with $a = -1$ to get

$$I = \int_{-1}^{1} \frac{dx}{1 - x^5 + \sqrt{1 + x^{10}}}$$

Adding the two versions of I together then gives:

$$2I = \int_{-1}^{1} \frac{(1 - x^5 + \sqrt{1+x^{10}}) + (1 + x^5 + \sqrt{1+x^{10}})}{((1 - x^5 + \sqrt{1+x^{10}})(1 + x^5 + \sqrt{1+x^{10}})}$$

$$\Rightarrow I = \int_{-1}^{1} \frac{1 + \sqrt{1+x^{10}}\,dx}{1 - x^{10} + 1 + x^{10} + (1 - x^5 + 1 + x^5)(\sqrt{1+x^{10}})}$$

$$= \int_{-1}^{1} \frac{1 + \sqrt{1+x^{10}}dx}{2 + 2\sqrt{1+x^{10}}} = \int_{-1}^{1} \frac{dx}{2} = 2 \times \frac{1}{2} = 1.$$

10. From the substitution $x = \frac{4}{t}$ we get $dx = -\frac{4dt}{t^2}$ so that the integral I becomes:

$$I = -\int_{\infty}^{0} \frac{\ln 4 - \ln t}{\frac{16}{t^2} + \frac{8}{t} + 4} \cdot \frac{4}{t^2}\, dt = \ln 4 \int_0^{\infty} \frac{dt}{t^2 + 2t + 4} - I$$

$$\Rightarrow 2I = \ln 4 \int_0^\infty \frac{dt}{(t+1)^2+3} \Rightarrow I = \frac{\ln 4}{2} \int_1^\infty \frac{du}{u^2+(\sqrt{3})^2}$$

$$\Rightarrow I = \frac{\ln 4^{\frac{1}{2}}}{\sqrt{3}} \big[\arctan(\frac{u}{\sqrt{3}})\big]_1^\infty = \frac{\ln 2}{\sqrt{3}} \big[\arctan(\infty) - \arctan(\frac{1}{\sqrt{3}})\big]$$

$$= \frac{\ln 2}{\sqrt{3}}[\frac{\pi}{2} - \frac{\pi}{6}] = \frac{\ln 2}{\sqrt{3}}(\frac{\pi}{3}) = \frac{\pi\sqrt{3}\ln 2}{9} \approx 0 \cdot 419.$$

Problem Set 6 Separable and linear first order differential equations

1. A separable equation:

$$\frac{dy}{y^2} = -\frac{dt}{t} \Rightarrow -y^{-1} = -\ln|t| + C$$

$$\Rightarrow y = (\ln|t| + C)^{-1}.$$

2. Again separating variables:

$$\frac{dy}{1+y^2} = e^x\,dx \Rightarrow \arctan y = e^x + C$$

$$\Rightarrow y = \tan(e^x + C).$$

3. The information translates as the d.e.

$$\frac{dy}{dx} = \frac{1}{y} \Rightarrow \int y\,dy = \int dx$$

$$\Rightarrow \frac{1}{2}y^2 = x + C \Rightarrow y = \sqrt{2x+C}.$$

4. Similary to Question 3 we obtain:

$$\frac{dy}{dx} = y^2 \Rightarrow \int \frac{dy}{y^2} = \int dx$$

$$\Rightarrow -\frac{1}{y} = x + C \Rightarrow y = -\frac{1}{x+C}.$$

5. We have $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$ and our equation becomes

$$v + x\frac{dv}{dx} = \frac{vx-4x}{x-vx} = \frac{v-4}{1-v} \Rightarrow \frac{dv}{dx} = \frac{1}{x}\big(\frac{v-4}{1-v} - v\big) =$$

$$\frac{1}{x} \cdot \frac{v-4-v(1-v)}{1-v} = \frac{1}{x} \cdot \frac{v^2-4}{1-v} \Rightarrow$$

$$\frac{dx}{x} = \frac{1-v}{v^2-4}dv \tag{1}$$

and so the equation is separable in v and x. Now

$$\frac{1-v}{v^2-4} = \frac{1-v}{(v-2)(v+2)} = -\frac{1}{4(v-2)} - \frac{3}{4(v+2)}.$$

Integrating both sides of the expression in (1) gives

$$\ln x = -\frac{1}{4}\big(\ln|v-2| + 3\ln|v+2|)\big) + c \Rightarrow \ln|x^{-4}| = \ln|(v-2)(v+2)^3| + c \Rightarrow$$

$$|x^{-4}| = A|(\frac{y}{x}-2)(\frac{y}{x}+2)^3| = A|\frac{y-2x}{x}\cdot\frac{(y+2x)^3}{x^3}| \Rightarrow$$

$$|(y-2x)(y+2x)^3| = C, \text{ a positive constant.}$$

6. Homogeneous equation: put $y = vx$. Then $\frac{dy}{dx} = v + x\frac{dv}{dx}$ and our equation becomes:

$$v + x\frac{dv}{dx} = -\frac{1}{2}\big(v^{-1} + v\big) = -\frac{1+v^2}{2v}$$

$$\Rightarrow x\frac{dv}{dx} == -\frac{1+v^2+2v^2}{2v} = -\frac{3v^2+1}{2v}$$

$$\Rightarrow -\frac{2v\,dv}{3v^2+1} = \frac{dx}{x}$$

$$\Rightarrow -\frac{1}{3}\ln(1+3v^2) = \ln|x| + C$$

$$\Rightarrow \ln|x|(1+3v^2)^{\frac{1}{3}} = C$$

$$\Rightarrow x^3(1+3v^2) = C \Rightarrow x^3(1+\frac{3y^2}{x^2}) = C$$

$$\Rightarrow x^3 + 3xy^2 = C.$$

Applying the initial condition that $y(1) = 1$ gives $1^3 + 3(1)(1) = C$ so that

$$x^3 + 3xy^2 = 4.$$

7. Exact equation as with $M(x,y) = x^2 + y^2$ and $N(x,y) = 2xy + \cos y$ we have $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 2y$. We therefore seek a solution $f(x,y) = C$ where $\frac{\partial f}{\partial x} = x^2 + y^2$ and $\frac{\partial f}{\partial y} = 2xy + \cos y$ giving

$$f(x,y) = \frac{x^3}{3} + xy^2 + g(y) \Rightarrow \frac{\partial f}{\partial y} = 2xy + \frac{dg}{dy} = 2xy + \cos y$$

$$\Rightarrow g(y) = \sin y \text{ and so our solution is:}$$

$$f(x,y) = \frac{x^3}{3} + xy^2 + \sin y = C.$$

8. We seek $f(x,y)$ such that

$$f_x = y\cos x + 2xe^y \Rightarrow f = y\sin x + x^2e^y + g(y),$$

we then obtain $f_y = \sin x + x^2e^y + g'(y) = \sin x + x^2y + 2$ whence $f(x,y) = y\sin x + x^2e^y + 2y$ (the constant of integration may be omitted, or what is the same, taken as zero). Hence our solution is

$$y\sin x + x^2e^y + 2y = C.$$

9. This is not exact as $\frac{\partial(2y)}{\partial y} = 2$ and $\frac{\partial x}{\partial x} = 1$. Multiplying by the integrating factor of x gives $2xydx + x^2dy = 0$ and $\frac{\partial(2xy)}{\partial y} = 2x = \frac{\partial(x^2)}{\partial x}$. Putting $\frac{\partial f}{\partial x} = 2xy$ gives $f(x,y) = x^2y + g(y)$ so that $\frac{\partial f}{\partial y} = x^2 + \frac{dg}{dy} = x^2$ so that we may take $g(y) = 0$ and for our general solution we obtain:

$$x^2y = C \text{ or } y = \tfrac{C}{x^2}.$$

10. $\mu = (xy^2)^{-1}$ is an integrating factor for multiplying through by μ gives the equation:

$$(\frac{1}{x} + \frac{1}{y})dx - \frac{x}{y^2}dy = 0.$$

To check: $M(x,y) = \frac{1}{x} + \frac{1}{y} \Rightarrow M_y = -\frac{1}{y^2}$ and $N(x,y) = -\frac{x}{y^2} \Rightarrow N_x = -\frac{1}{y^2}$, thus showing exactness. To find $f(x,y)$ we next write

$$f_x = M = \frac{1}{x} + \frac{1}{y} \Rightarrow f(x,y) = \ln|x| + \frac{x}{y} + g(y); \text{ hence}$$

$$f_y = -\frac{x}{y^2} + \frac{dg}{dy} = -\frac{x}{y^2}$$

so we may take $g(y) = 0$ to obtain the implicit solution to our differential equation:

$$\ln|x| + \frac{x}{y} = c;\ x \neq 0, y \neq 0.$$

Comment: Here we relied on a rabbit-out-of-the-hat integrating factor but sometimes we can be systematic. For instance, if $(M_y - N_x)/N$ is a function of x only then the equation can be made exact by multiplying by the integrating factor μ, which is itself the solution of the linear differential equation

$$\frac{d\mu}{dx} = \frac{M_y - N_x}{N}\mu.$$

For example, this method will yield that $\mu = x$ is an integrating factor of $(3xy + y^2)dx + (x^2 + xy)dy = 0$, the solution of which is then found to be $x^3y + \frac{1}{2}x^2y^2 = c$.

Problem Set 7 Linear and second order differential equations

1. We have $\frac{dy}{dx}+\frac{3}{x}y = x$, which is a first order linear equation with integrating factor $\rho(x) = e^{\int \frac{3\,dx}{x}} = e^{3\ln x} = e^{\ln x^3} = x^3$. Multiplying throughout by $\rho(x)$ gives

$$x^3\frac{dy}{dx} + 3x^2y = x^4 \Rightarrow (x^3y)' = x^4$$

$$\Rightarrow x^3y = \frac{x^5}{5} + C$$

$$\therefore y = \frac{x^2}{5} + \frac{C}{x^3}.$$

Comment: as with any equation, we can check that the solution works:

$$x\frac{dy}{dx} + 3y = x\frac{2x}{5} - 3x\frac{C}{x^4} + \frac{3x^2}{5} + \frac{3C}{x^3} = x^2.$$

2. Another linear first order equation, $\frac{dy}{dx} + y\tanh x = \frac{2}{e^x(e^x+e^{-x})}$ with $\rho(x) = e^{\int \frac{\sinh x\,dx}{\cosh x}} = e^{\ln(\cosh x)} = \cosh x$ so we obtain:

$$\cosh x\frac{dy}{dx} + y\sinh x = e^{-x}$$

$$\Rightarrow (y\cosh x)' = e^{-x} \Rightarrow y\cosh x = C - e^{-x}$$

$$\Rightarrow y = -\frac{C - 2e^{-x}}{e^x - e^{-x}} = \frac{Ce^x - 2}{e^{2x} - 1}.$$

3. We have the linear equation $\frac{di}{dt} + \frac{Ri}{L} = \frac{V}{L}$ so that $\rho(t) = e^{\int \frac{R\,dt}{L}} = e^{\frac{Rt}{L}}$. Hence we have

$$(e^{\frac{Rt}{L}}i)' = \frac{V}{L}e^{\frac{Rt}{L}} \Rightarrow e^{\frac{Rt}{L}}i = C + \frac{V}{R}e^{\frac{Rt}{L}}.$$

Using the initial condition that $i(0) = 0$ we obtain $0 = C + \frac{V}{R}$ so that $C = -\frac{V}{R}$:

$$\Rightarrow e^{\frac{Rt}{L}}i(t) = \frac{V}{R}(e^{\frac{Rt}{L}} - 1)$$

$$\therefore i(t) = \frac{V}{R}(1 - e^{-\frac{Ri}{L}}).$$

4. $\rho(x) = e^{\int -2x\,dx} = e^{-x^2}$ so that

$$(ye^{-x^2})' = xe^{-x^2} \Rightarrow ye^{-x^2} = \int xe^{-x^2}\,dx = c - \frac{1}{2}e^{-x^2}$$

$$\Rightarrow y = ce^{x^2} - \frac{1}{2};\ \text{put } y(0) = 1;$$

$$1 = ce^0 - \frac{1}{2} \Rightarrow c = \frac{3}{2};$$
$$\therefore y = \frac{3}{2}e^{x^2} - \frac{1}{2}.$$

5. The auxiliary equation is $\lambda^2 - 3\lambda - 4 = (\lambda - 4)(\lambda + 1) = 0 \Rightarrow \lambda = 4$ or $\lambda = -1$. Hence the general solution is $y(x) = Ae^{4x} + Be^{-x}$.

We put $y_p(x) = A\cos x + B\sin x$ so that $y_p'(x) = -A\sin x + B\cos x$ and $y_p''(x) = -A\cos x - B\sin x$. Substituting accordingly into our equation and collecting like terms gives;

$$(-A - 3B - 4A)\cos x + (-B + 3A - 4B)\sin x = 2\sin x$$
$$\Rightarrow -5A - 3B = 0 \,\&\, 3A - 5B = 2 \Rightarrow A = \frac{3}{17},\ B = -\frac{5}{17}$$
$$\Rightarrow y_p(x) = \frac{1}{17}\big(3\cos x - 5\sin x\big).$$

Hence the general solution to our equation is

$$y = Ae^{4x} + Be^{-x} + \frac{1}{17}(3\cos x - 5\sin x) \Rightarrow y' = 4Ae^{4x} - Be^{-x} - \frac{1}{17}(3\sin x + 5\cos x).$$

Putting $y(0) = y'(0) = 1$ gives the equations $A + B + \frac{3}{17} = 1,\ 4A - B - \frac{5}{17} = 1$. Adding these equations gives $5A = \frac{36}{17} \Rightarrow A = \frac{36}{85}$, whence $B = 1 - \frac{3}{17} - \frac{36}{85} = \frac{2}{5}$. Hence the solution required is:

$$y = \frac{36}{85}e^{4x} + \frac{2}{5}e^{-x} + \frac{3}{17}\cos x - \frac{5}{17}\sin x.$$

6. We seek a particular solution as a polynomial of the same degree, that is put $y_p(x) = Ax^2 + Bx + C \Rightarrow y_p'(x) = 2Ax + B,\ y_p''(x) = 2A$. Substituting accordingly gives the equation

$$2A - 3(2Ax + B) - 4(Ax^2 + Bx + C) = 4x^2$$
$$\Rightarrow -4Ax^2 - (4B + 6A)x + (-4C - 3B + 2A) = 4x^2.$$

Equating coefficients gives

$$A = -1,\ B = -\frac{3}{2}A = \frac{3}{2} \text{ and}$$
$$C = \frac{2A - 3B}{4} = \frac{1}{4}\big(-2 - \frac{9}{2}\big) = -\frac{13}{8}.$$
$$\text{Therefore } y_p(x) = -x^2 + \frac{3}{2}x - \frac{13}{8}.$$

.

7. From Question 5 we see that e^{-x} is a solution to the homogeneous equation so our form of particular solution is $y_p(x) = Axe^{-x}$, whence

$$y_p'(x) = Ae^{-x} - Axe^{-x},\ y_p''(x) = -2Ae^{-x} + Axe^{-x}.$$

Substituting accordingly gives:

$$(-2Ae^{-x} + Axe^{-x}) - 3(Ae^{-x} - Axe^{-x}) - 4(Axe^{-x}) = e^{-x}$$

$$\Rightarrow -5Ae^{-x} = e^{-x} \Rightarrow A = -\frac{1}{5}.$$

Hence a particular solution is

$$y_p(x) = -\frac{1}{5}xe^{-x}.$$

8. Putting $p = \frac{dy}{dx}$ gives $x\frac{dp}{dx} + p = x^2$ so that $\frac{dp}{dx} + \frac{p}{x} = x$. Hence $\rho(x) = e^{\int \frac{dx}{x}} = e^{\ln x} = x$. Therefore

$$(px)' = x^2 \Rightarrow px = \frac{x^3}{3} + C$$

$$\therefore p = \frac{dy}{dx} = \frac{x^2}{3} + \frac{C}{x} \Rightarrow y(x) = \frac{x^3}{9} + C\ln|x| + D.$$

Substituting $y = 0$ and $y' = 1$ when $x = 1$ into the solution gives, $0 = \frac{1}{9} + 0 + D$ so that $D = -\frac{1}{9}$ and $1 = \frac{1}{3} + C$ so that $C = \frac{2}{3}$, which all yields:

$$y(x) = \frac{x^3}{9} + \frac{2}{3}\ln|x| - \frac{1}{9}.$$

9. Differentiating $x_1' = x_1 + x_2$ and substituting using the given equations gives

$$x_1'' = x_1' + x_2' = x_1' + (4x_1 + x_2)$$

$$\Rightarrow x_1'' = x_1' + 4x_1 + (x_1' - x_1) \Rightarrow x_1'' - 2x_1' - 3x_1 = 0.$$

The characteristic equation of this d.e. is $\lambda^2 - 2\lambda - 3 = (\lambda + 1)(\lambda - 3) = 0$ so that

$$x_1(t) = Ae^{-t} + Be^{3t} \Rightarrow x_1'(t) = -Ae^{-t} + 3Be^{3t}$$

$$\Rightarrow x_2(t) = x_1'(t) - x_1(t) = -Ae^{-t} + 3Be^{3t} - Ae^{-t} - Be^{3t}$$

$$= -2Ae^{-t} + 2Be^{3t}.$$

$$\therefore x_1(t) = Ae^{-t} + Be^{3t},\ x_2(t) = -2Ae^{-t} + 2Be^{3t}.$$

10. Writing $\mathbf{x} = (x_1(t), x_2(t))^T$ we may express the system in the form $\mathbf{x}' = A\mathbf{x}$ where

$$A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix},$$

with characteristic equation

$$(1-\lambda)^2 = 4 \Rightarrow 1 - \lambda = \pm 2 \Rightarrow \lambda \in \{-1, 3\};$$

taking $\lambda = -1$ we get for an eignevector the equation $2x + y = 0$ so that $\mathbf{e}_1 = (1, -2)^T$ while for $\lambda = 3$ we have $-2x+y = 0$ so we may take $\mathbf{e}_2 = (1, 2)^T$. Our general solution is then:

$$\mathbf{x} = Ae^{\lambda_1 t}\mathbf{e}_1 + Be^{\lambda_2 t}\mathbf{e}_2 = Ae^{-t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} + Be^{3t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\Leftrightarrow x_1(t) = Ae^{-t} + Be^{3t},\ x_2(t) = -2Ae^{-t} + 2Be^{3t}.$$

Comment Note that the characteristic equation of the second order differential equation in Question 9 is the same as the characteristic equation of the coefficient matrix in Question 10.

Problem Set 8 Power Series

1. The series has the form

$$\sum_{n=0}^{\infty} (\frac{2}{3})^n \frac{1}{n^2+1}(x + \frac{5}{2})^n$$

so that the centre of convergence is $x = -\frac{5}{2}$. The radius of convergence R satisfies

$$\frac{1}{R} = L = \lim_{n\to\infty} |\frac{\left(\frac{2}{3}\right)^{n+1}\frac{1}{(n+1)^2+1}}{\left(\frac{2}{3}\right)^n\frac{1}{n^2+1}}| = \lim_{n\to\infty} \frac{2}{3}\frac{n^2+1}{(n+1)^2+1} = \frac{2}{3}.$$

Thus $R = \frac{3}{2}$. The series then converges absolutely on $(-\frac{5}{2} - \frac{3}{2}, -\frac{5}{2} + \frac{3}{2}) = (-4, -1)$. At the endpoints of $x = -4$ and $x = -1$ the respective series are

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2+1},\ \sum_{n=0}^{\infty} \frac{1}{n^2+1},$$

which both converge. Therefore the interval of convergence of the power series is $[-4, -1]$.

2. Integrating term-by-term we find that

$$\arctan x = \int_0^x \frac{dt}{1+t^2} = \sum_{n=0}^{\infty}\int_0^x (-1)^n t^{2n} dt = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1}$$

$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + \frac{(-1)^n x^{2n+1}}{2n+1} + \cdots,\ -1 < x < 1.$$

3.

$$\ln|1+x| = \int_0^x \frac{dt}{1+t} = \sum_{n=0}^{\infty}\int_0^x (-1)^n t^n dt = \sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{n+1},\ -1 < x < 1;$$

replacing x by $-x$ in the previous formula gives:

$$\ln|1-x| = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}.$$

Summing these formulas and dividing by 2 then gives on the interval $(-1,1)$:

$$\frac{1}{2}\ln|\frac{1+x}{1-x}| = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1} = x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots .$$

4. The pattern of derivatives of $\sin x$, beginning from the 0th derivative, is $\sin x, \cos x, -\sin x, -\cos x, \sin x, \cdots$, a cycle of length 4. Hence the terms $f^{(n)}(0)$ in this case have the pattern $\sin 0, \cos 0, -\sin 0, -\cos 0, \cdots$, which is to say $0, 1, 0, -1, \cdots$ .It follows that the McLaurin series for $f(x) = \sin x$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots . \tag{2}$$

which, by the ratio test converges for all $x \in \mathbb{R}$:

$$\frac{1}{R} = \lim_{n\to\infty} |\frac{(-1)^{n+1}x^{2n+3}}{(2n+3)!} \cdot \frac{(2n+1)!}{(-1)^n x^{2n+1}}| = \lim_{n\to\infty} |\frac{x^2}{(2n+3)(2n+2)}| = 0.$$

Since this series alternates in sign and the absolute value of the terms decreases montonically to 0, the error in truncating the series at the nth term, a_n is bounded by $|a_{n+1}|$. We wish to estimate the value of $\sin 3^\circ = \sin \frac{3\pi}{180} = \sin \frac{\pi}{60}$. For $n = 2$ in (2) we find that term has absolute value

$$\frac{\pi^5}{60^5 \cdot 5!} < 0 \cdot 5 \times 10^{-5}.$$

Hence our required approximate value for $\sin 3^\circ$ is given by

$$\frac{\pi}{60} - \frac{\pi^3}{60^3 \cdot 3!} \approx 0 \cdot 05234.$$

5.

$$\begin{aligned} E(x) &= \int_0^x (1 - t^2 + \frac{t^4}{2!} - \frac{t^6}{3!} + \frac{t^8}{4!} - \cdots)\,dt \\ &= (t - \frac{t^3}{3} + \frac{t^5}{5\times 2!} - \frac{t^7}{7\times 3!} + \frac{t^9}{9\times 4!} - \cdots)|_0^x \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)n!}, \end{aligned}$$

and this series converges for all x as this is true of the series for e^x. Again, the error does not exceed the first omitted term so, with $x = 1$, we need the least n such that,

$$\frac{1}{(2n+1)n!} < 0 \cdot 0005 \Leftrightarrow (2n+1)n! > 2,000.$$

By trial we find that $13 \times 6! = 9,360$ we suffice. Thus we take

$$E_n(x) = \frac{f^{(n+1)}(X)}{(n+1)!}(x - x_0)^{n+1}$$

$$E(1) \approx 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} - \frac{1}{1.320} \approx 0 \cdot 747.$$

6. We apply the error bound $E_n(x) = \frac{f^{(n+1)}(X)}{(n+1)!}(x - x_0)^{n+1}$; here $x_0 = 0$, $f^{(n+1)}(x) = e^X \leq e^1 = e$ and so we seek the least n such that

$$\frac{e}{(n+1)!} < 10^{-6} \Leftrightarrow (n+1)! > e \times 10^6;$$

we find that $10! = 3,628,800$ is the least factorial that exceed our bound, giving $n = 9$ gives the required approximation, so the final term required in the sum is $\frac{1}{9!}$.

7. With $f(x) = (1+x)^r$ we see that for $n \geq 1$, $f^{(n)}(x) = r(r-1)\cdots(r-n+1)(1+x)^{r-n}$ so the binomial series is

$$(1+x)^r = 1 + \sum_{n=1}^{\infty} \frac{r(r-1)(r-2)\cdots(r-n+1)}{n!}x^n.$$

To apply the ratio test we seek the limit

$$\lim_{n\to\infty} \left| \frac{\frac{r(r-1)\cdots(r-n)}{(n+1)!}x^{n+1}}{\frac{r(r-1)\cdots(r-n+1)}{n!}x^n} \right| = \lim_{n\to\infty} \left| \frac{r-n}{n+1} \right| = |x|,$$

so that the series converges if $-1 < x < 1$.

8. Putting $r = -\frac{1}{2}$ in the binomial series gives the expansion

$$\frac{1}{\sqrt{1+x}} = 1 + \sum_{n=1}^{\infty} \frac{(-\frac{1}{2})(-\frac{3}{2})\cdots(-\frac{2n-1}{2})}{n!}x^n$$

$$= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n 1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n n!}x^n;$$

this series converges for $-1 < x \leq 1$ (for $x = 1$, series converges by alternating series test).

9. Replacing x by $-t^2$ in the series of Question 8 and then integrating we get

$$\frac{1}{\sqrt{1-t^2}} = 1 + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n n!}t^{2n};$$

$$\sin^{-1} x = \int_0^x \frac{dt}{\sqrt{1-t^2}} = x + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n n!(2n+1)}x^{2n+1}$$

$$= x + \frac{x^3}{6} + \frac{3}{40}x^5 + \cdots \ (-1 < x < 1).$$

10. We assume that

$$e^{ix} = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{4n}}{(4n)!} - \frac{x^{4n+2}}{(4n+2)!} + i\Big(\sum_{n=0}^{\infty} \frac{x^{1+4n}}{(1+4n)!} - \frac{x^{3+4n}}{(3+4n)!}\Big);$$

taking the real part of this equation then gives

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} - \cdots, \ x \in \mathbb{R}.$$

Problem Set 9 Functions of several variables

1. The point P is indeed on the given circle, as is readily verified and in the upper half sphere, which is described by the equation $z = \sqrt{1 - x^2 - y^2}$. Then

$$\frac{\partial z}{\partial y} = \frac{\partial((1 - x^2 - y^2)^{\frac{1}{2}})}{\partial y} = -\frac{y}{\sqrt{1 - x^2 - y^2}} \text{ and we require}$$

$$\frac{\partial z}{\partial y}|_{x=\frac{2}{3}, y=\frac{1}{3}} = -\frac{\frac{1}{3}}{\sqrt{1 - (\frac{2}{3})^2 - (\frac{1}{3})^2}} = -\frac{1}{2}.$$

Comment Alternatively, differentiate $x^2 + y^2 + z^2 = 1$ implicitly with respect to y with x held constant to get $2y + 2z\frac{\partial z}{\partial y} = 0$ so that $\frac{\partial z}{\partial y} = -\frac{y}{z}$; evaluating this at P then gives the same answer.

2. We have $P = kTV^{-1}$ so that

$$\frac{\partial P}{\partial V} = -\frac{kT}{V^2} = -\frac{1}{V} \cdot \frac{kT}{V} = -\frac{P}{V}.$$

3. $f(x, y) = e^{2x-y} + \sin xy$ so that

$$f_x = 2e^{2x-y} + y\cos xy \Rightarrow f_{yx} = -2e^{2x-y} + \cos xy - xy\sin xy;$$

$$f_y = -e^{2x-y} + x\cos xy \Rightarrow f_{xy} = -2e^{2x-y} + \cos xy - xy\sin xy = f_{yx}.$$

4. We see that $f_x = \frac{2x}{x^2+y^2}$ so that

$$f_{xx} = \frac{2(x^2 + y^2) - 4x^2}{(x^2 + y^2)^2} = \frac{2y^2 - 2x^2}{(x^2 + y^2)^2};$$

By symmetry it follows that

$$f_{yy} = \frac{2x^2 - 2y^2}{(x^2 + y^2)^2}$$

$$\Rightarrow f_{xx} + f_{yy} = 0.$$

5. We have

$$\frac{\partial u}{\partial x} = \frac{2x}{x^2+y^2}, \ \frac{\partial v}{\partial y} = \frac{\frac{2}{x}}{1+(\frac{y^2}{x^2})} = \frac{2x}{x^2+y^2}, \text{ in accord with the first equation.}$$

$$\frac{\partial u}{\partial y} = \frac{2y}{x^2+y^2}, \ -\frac{\partial v}{\partial x} = -\frac{2(\frac{-y}{x^2})}{1+\frac{y^2}{x^2}} = \frac{2y}{x^2+y^2}, \text{ in accord with the second equation.}$$

6.

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} = (y)(-\sin t) + (x)(\cos t) + (1)(1)$$

$$= (\sin t)(-\sin t) + (\cos t)(\cos t) = 1 = -\sin^2 t + \cos^2 t + 1 = 1 + \cos 2t.$$

$$\frac{dw}{dt}|_{t=0} = 1 + \cos(0) = 2.$$

7.

$$\frac{\partial z}{\partial u} = (ye^{xy})(2) + (xe^{xy})(\frac{1}{v}) = (2y + \frac{x}{v})e^{xy}$$

$$= (\frac{2u}{v} + \frac{2u+v}{v})e^{(2u+v)(\frac{u}{v})} = \big(1 + \frac{4u}{v}\big)e^{(2u+v)(u/v)}.$$

$$\frac{\partial z}{\partial v} = (ye^{xy})(1)(xe^{xy})\big(-\frac{u}{v^2}\big) = \big(y - x\big(\frac{u}{v^2}\big)\big)e^{xy}$$

$$= \big(\frac{u}{v} - (2u+v)\big(\frac{u}{v^2}\big)e^{(2u+v)(u/v)} = -\frac{2u^2}{v^2}e^{(2u+v)(u/v)}.$$

8.

$$dT = \frac{\partial T}{\partial L}dL + \frac{\partial T}{\partial g}dg = \frac{2\pi}{2\sqrt{Lg}}dL - \frac{2\pi}{2g^{\frac{3}{2}}}dg.$$

We have $dL = \frac{2}{100}L$ and $dg = -\frac{6}{1000}g$. Thus

$$dT = \frac{1}{100}2\pi\sqrt{\frac{L}{g}} - \big(-\frac{6}{1000}\big)\frac{2\pi}{2}\sqrt{\frac{L}{g}} = \frac{13}{1000}T.$$

Therefore the period T of the pendulum increases by 1.3%.

9.

$$\nabla f = \big(\frac{(x^2+y^2)-2x^2}{(x^2+y^2)^2}, \frac{-2x^2}{(x^2+y^2)^2}\big) = \big(\frac{y^2-x^2}{(x^2+y^2)^2}, -\frac{2x^2}{(x^2+y^2)^2}\big).$$

Hence $\nabla f(2,3) = \big(\frac{3^2-2^2}{(2^2+3^2)^2}, -\frac{2(2^2)}{(2^2+3^2)^2}\big) = \big(\frac{5}{169}, -\frac{8}{169}\big)$.

10. At the point $(x_0, y_0) = (2,3)$ we have $z_0 = \frac{2}{2^2+3^2} = \frac{2}{9}$; $f_x(x_0,y_0) = \frac{5}{169}$, $f_y(x_0,y_0) = -\frac{8}{169}$. The equation of the tangent plane at (x_0,y_0,z_0) is

$$f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) = z - z_0, \text{ which gives:}$$

$$\frac{5}{169}(x-2)-\frac{8}{169}(y-3)=z-\frac{2}{9}.$$

Problem Set 10

1. $\nabla f(x,y)=(6xy,3x^2)$; $\nabla f\big|_{(1,2)}=(6(1)(2),3(1^2))=(12,3)$; next $|\mathbf{u}|=\sqrt{3^2+4^2}=5$ and so $\hat{\mathbf{u}}=\frac{1}{5}\mathbf{u}$. Hence

$$D_{\mathbf{u}}f(1,2)=\frac{1}{5}(12,3)\bullet(3,4)=\frac{1}{5}(36+12)=\frac{48}{5}.$$

2. $\nabla T(x,y)=(2xe^{-y},-x^2e^{-y})$. The temperature increases most rapidly in the direction of the gradient, which at $(2,1)$ has value $\nabla T(1,2)=(2e^{-2},-e^{-2})$; equivalently the direction of $(2,-1)$.

3. The tangent plane equation is $\nabla f\big|_{(x_0,y_0)}\bullet(\mathbf{x}-\mathbf{x_0})=z-z_0$, where $\mathbf{x}=(x,y)$, $\mathbf{x_0}=(x_0,y_0)$. We have

$$\nabla f\big|_{(2,1)}=(2xy,x^2)\big|_{(2,1)}=\big((2(2)(1),2^2\big)=(4,4).$$

Hence our tangent plane is given by

$$(4,4)\bullet\big((x-2),(y-1)\big)=z-4\Rightarrow 4(x-2)+4(y-1)=z-4\Rightarrow$$

$$4x+4y-z=8.$$

The vector $(4,4,1)$ is normal to this plane so the normal line at $(2,1,4)$ has equation:

$$\frac{x-2}{4}=\frac{y-1}{4}=z-4.$$

4. We have $f_x(x,y)=2x-4y+12$, $f_y(x,y)=-4x-4y-12$. For a horizontal tangent plane at (x_0,y_0) we must have both partial derivatives equal 0, thus we obtain the equations:

$$(x-2y=-6,\ \&\, x+y=-3)\Rightarrow -3y=-3\Rightarrow y=1,\ x=-3-1=-4.$$

Hence $(x_0,y_0)=(-4,1)$. The corresponding value of $z=z_0$ on the surface satisfies $z_0=$

$$(-4)^2-4(-4)(1)-2(1^2)+12(-4)-12(1)-1=16+16-2-48-12-1=-31.$$

Hence the point of tangency is $(-4,1,-31)$. The equation of the tangent plane is $z=-31$.

5.

$$f_x(x,y)=3x^2-x-2,\ f_y(x,y)=9y^2-9.$$

For stationary points, we put $f_x = f_y = 0$ giving

$$3x^2 - x - 2 = (3x+2)(x-1) = 0 \Rightarrow x = 1, -\frac{2}{3}$$

and $y^2 = 1 \Rightarrow y = \pm 1$. Hence our four stationary points on the surface form the set

$$\{(1,1),(1,-1),(-\frac{2}{3},1),(-\frac{2}{3},-1)\}.$$

6. We have

$$f_{xx}(x,y) = 6x - 1,\ f_{yy}(x,y) = 18y,\ f_{xy}(x,y) = 0.$$

Hence $D(x,y) = f_{xx}f_{yy} - (f_{xy})^2 = f_{xx}f_{yy}$. At the respective points as listed in Q8 we obtain:

$$D(1,1) = (6-1)(18) > 0,\ f_{xx}(1,1) > 0 \Rightarrow (1,1)\,\text{is a minimum;}$$

$$D(1,-1) = (6-1)(-18) < 0 \Rightarrow (1,-1)\,\text{is a saddle point;}$$

$$D(-\frac{2}{3},1) = (-4-1)(18) < 0 \Rightarrow (-\frac{2}{3},1)\,\text{is a saddle point,}$$

$$D(-\tfrac{2}{3},-1) = (-4-1)(-18) > 0,\ f_{xx}(-\tfrac{2}{3},-1) < 0 \Rightarrow (-\tfrac{2}{3},-1)\,\text{is a maximum.}$$

7. We minimize the square of the distance of the point $(-1,3,2)$ to points (x,y,z) where $z = 4 - x + 2y$. This gives the function

$f(x,y) = (x+1)^2 + (y-3)^2 + (4-x+2y-2)^2 = 2x^2 + 5y^2 - 4xy - 2x + 2y + 14$. Hence

$$f_x(x,y) = 4x - 4y - 2,\ f_y(x,y) = 10y - 4x + 2.$$

Put $f_x = f_y = 0$ to obtain:

$$2x - 2y = 1 = 2x - 5y \Rightarrow y = 0,\ x = \frac{1}{2},\ z = \frac{7}{2}.$$

Therefore the required closest point is $(\frac{1}{2}, 0, \frac{7}{2})$.

Comment Note that $f_{xx}f_{yy} - (f_{xy}) = 4 \times 4 - (-4)^2 = 0$ is inconclusive in this case but the point must represent a minimum as it is the extreme point of a positive quantity that becomes unbounded far from the point from which we are measuring. The geometric alternative is to find the intersection of the plane with the line through $(-1,3,2)$ in the direction of a normal vector to the plane, which we can take to by $\mathbf{u} = (1,-2,1)$. This gives the line $x + 1 = -\frac{1}{2}(y-3) = z - 2$, which yields $z = x + 3$ and we also have $z = 4 + 2y - x$, which quickly gives the required point $(\frac{1}{2}, 0, \frac{7}{2})$ also.

8.

$$\nabla f(x,y) = (3x^2y^5, 5x^3y^4) = \lambda \nabla g(x,y) = (1,1).$$

Thus

$$3x^2y^5 = \lambda = 5x^3y^4, \Rightarrow 3y = 5x;\ \text{also}\ x + y = 8;$$

$$\Rightarrow x + \frac{5}{3}x = 8 \Rightarrow \frac{8}{3}x = 8 \Rightarrow x = 3,\ y = 8 - x = 8 - 3 = 5.$$

This gives the extreme point $P(x, y) = (3, 5)$. Since $f(x, y)$ is negative on the line except for the interval $0 \leq x \leq 8$ we see that P is indeed a maximum of f on the line $x + y = 8$.

9. Here we have, with an obvious notation, $xyz = 32$ so that $g(x, y, z) = xyz - 32 = 0$ is our constraint function; summing the area of the five sides of the open box we obtain the area function $f(x, y, z) = 2xy + 2yz = xy$. Hence we put

$$\nabla f(x, y, z) = (2z + y, 2z + x, 2x + 2y) = \lambda \nabla g(x, y, z) = \lambda(yz, wz, xy).$$

Hence

$$2z + y = \lambda yz,\ 2z = x = \lambda xz,\ 2(x + y) = \lambda xy \Rightarrow$$

$$\frac{2}{y} + \frac{1}{z} = \frac{2}{x} + \frac{1}{z} = \frac{2}{x} + \frac{2}{y} \Rightarrow \frac{2}{y} = \frac{2}{x} \Rightarrow x = y,\ z = \frac{y}{2} = \frac{x}{2}.$$

Substituting in $xyz = 32$ we get $x^3 = 64 \Rightarrow x = y = 4,\ z = 2$. The box with minimum surface area for a volume of 32cm^3 has dimensions $4 \times 4 \times 2$.

10. We wish to minimize $f(x, y) = x^2 + y^2$ subject to the constraint $g(x, y) = x^2 y - 16 = 0$. We get

$$2x = 2\lambda xy \Rightarrow x(1 - \lambda y) = 0$$

$$2y = \lambda x^2 \Rightarrow 2y - \lambda x^2 = 0$$

$$x^2 y = 16.$$

The first equation gives either $x = 0$ or $\lambda y = 1$. However $x = 0$ is inconsistent with $x^2 y = 16$. Hence $\lambda y = 1$. Multiplying the second equation by y gives

$$2y^2 = \lambda y x^2 = x^2 \Rightarrow x = \pm\sqrt{2}y,$$

and from $x^2 y = 16$ we now get $2y^3 = 16$ so that $y = 2$. This gives two candidate points: $(\pm 2\sqrt{2}, 2)$, both of which have distance to the origin of $\sqrt{8 + 4} = \sqrt{12} = 2\sqrt{3}$ units, which clearly do represent points of closest approach.