Mathematics 202 Combinatorics & Number Theory

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The first five problem sets in this module are about techniques to solve first linear and then quadratic equations in modular arithmetic. The solving of quadratic congruences is a surprisingly intriguing subject and we shall take one of the main theorems, the so called *Quadratic Reciprocity Theorem* for granted. This theorem relates the existence of solutions to the equations $x^2 \equiv p \pmod q$ and $x^2 \equiv q \pmod p$. It was thought to be true for many years before being proved by Gauss, in eight different ways. Problem Set 3 introduces the so called *RSA algorithm*, which is the basis of pretty much all internet cryptography. It completely relies on congruences based on large prime numbers and is the certainly the most major application of number theory to the everyday world. It is arguably the single most important application of mathematics that we have.

Set 6 is on the topic of countable and uncountable sets. It was proved by Cantor using his *Diagonal Argument* that the integers and the real numbers cannot be matched in one-to-one correspondence and this questin set explores various interesting collections and asks which ones have the same *cardinality* as the integers and which do not.

Set 7 introduces *difference equations*, which are the discrete analogoue of differential equations and the elementary ones presented here may be solved using parallel techniques.

Set 8 introduces both standard and *exponential generating functions* for combinatorial sequences and gives problems that can be solved using them. Set 9 is based on the *Inclusion-Exclusion Principle*, which is almost a common sense piece of mathematics that yet has powerful applications and is indispensable for many combinatorial counting problems. The final set introduces *Catalan numbers*, which count a number of different types of combinatorial objects and are intimately related to the *central binomial coefficients*.

Problem Set 1 Linear Congruences

Recall that the statement that integers a and b are *congruent modulo* n, written, $a \equiv b \pmod{n}$ means that $b = a + kn$ for some integer k. A *linear congruence* is an equation of the form $ax \equiv b \pmod{n}$. Find all *least residue solutions*, which means solutions in the range $0 \leq x \leq n-1$ for the following congruences. Let $d =$ gcd of a and n. There are no solutions to the congruence equation if d is not a factor of b but if it is then there are d solutions.

1. $3x \equiv 2 \pmod{6}$
2. $5x \equiv 2 \pmod{6}$
3. $4x \equiv 2 \pmod{6}$.
4. $6x \equiv 14 \pmod{31}$.
5. $15x \equiv 12 \pmod{57}$.
6. Find a number that leaves a remainder of 1 when divided by 2, a remainder of 2 when divided by 3 and of 3 when divided by 5.
7. Find the smallest odd $n > 3$ such that $3|n$, $5|n+2$, and $7|n+4$.
8. Solve the simultaneous linear congruences

$$x + 2y \equiv 3 \pmod{7},\ 3x + y \equiv 2 \pmod{6}.$$

9. How many possibilities are there for the number of solutions of a linear congruence modulo 20?

10. February 1968 had five Thursdays. How many other years up to and including 2100 will be so rich in February Thursdays?

Problem Set 2 Linear Diophantine Equations

In Questions 1-3 find all solutions in integers.

1.
$$2x + y = 2.$$

2.
$$15x + 16y = 17.$$

3.
$$15x + 18y = 17.$$

4. Solve in positive integers

$$7x + 15y = 51.$$

5. Solve in negative integers

$$6x - 15y = 51.$$

6. Solve in positive integers the simultaneous equations:

$$x + y + z = 31, \ x + 2y + 3z = 41.$$

7. Suppose that a collection of centipedes, scorpions, and worms contains 296 legs and 35 heads. How many worms are there?

8. A farmer sold her sheep for £180 each and her cows for £290 a piece, receiving £2890. How many cows did she sell?

9. When Ann is half as old as Mary will be when Mary is three times as old as Mary is now, Mary will be five times as old as Ann is now. Neither Ann nor Mary may vote. How old is Ann?

10. Andy and Bob put their collections of vinyl records up for sale on the internet with Andy selling 30 records and Bob 40. Each sold some of their records at £5 each and the rest at a common lower price in an integer number of pounds. And each received the same amount of money overall. What is the smallest amount that each could have received?

Problem Set 3 RSA Cryptography

Recall that the greatest common divisor of integers a and b is written as $d = (a, b)$. The *Euler ϕ-function* $\phi(n)$ is defined on the positive integers by $\phi(n) = |\{k \leq n : (k, n) = 1\}|$; it has the property that $a^{\phi(n)} \equiv 1 \pmod{n}$ for any $1 \leq a \leq n - 1$.

1. For a prime p, find the value of $\phi(p^m)$.
2. Given that $\phi(ab) = \phi(a)\phi(b)$ if $(a, b) = 1$, show that

$$\phi(k) = k(1 - \frac{1}{p_1})(1 - \frac{1}{p_2)}) \cdots (1 - \frac{1}{p_r})$$

where $p_1, p_2, \cdots, p_r$ are the distinct prime factors of k.

RSA cryptography algorithm Bob sends Alice a message, coded as a number M. Her *private key* is (p, q, d) where p, q are primes. Her *public key*, known to all, is $(n = pq, e)$ where $(e, \phi(n)) = 1$.

Here we take $p = 3$, $q = 11$ and $e = 7$.

3. Find n, and $\phi(n)$ in this case and show that $e = 7$ satisfies the previous criterion.
4. The number d satisfies $1 \leq d \leq \phi(n) - 1$ and $ed \equiv 1 \pmod{\phi(n)}$. Find d in this example.
5. Bob sends to Alice $M^e \pmod{n}$. Find Bob's transmission for $M = 6$.
6. Show that

$$M^{ed} \equiv M \text{ (mod n)}.$$

7. Show how Alice can now recover Bob's plaintext message, $M = 6$.
8. Let $(p, q) = (23, 47)$ and $e = 15$. Let $M = 77$. Show that Bob's transmission is now 646.
9. Show that in this case $d = 135$.
10. Recover Bob's plaintext message $M = 77$ for Alice.

Problem Set 4 Quadratic congruences

Throughout, let p and q denote odd primes.

1. By a completing the square argument, show that any quadratic congruence

$$Ax^2 + Bx + C \equiv 0 \pmod{p},\ A \neq 0,$$

can be reduced to one of the form $y^2 \equiv a \pmod{p}$.

2. Apply this approach to reduce $2x^2 + 3x + 1 \equiv 0 \pmod{7}$ to the form $y^2 \equiv a \pmod{7}$ and find the two least residue solutions.

3. Solve $3x^2 + x + 4 \equiv 0 \pmod{7}$.

4. Show that if p is not a factor of a then $x^2 \equiv a \pmod{p}$ has either no solutions or exactly two (least residue) solutions.

5. By using the corresponding property or the Euler ϕ–function (see Set 3) derive *Fermat's lemma*, that

$$a^p \equiv a \pmod{p}.$$

6. Deduce from Question 5 that

$$a^{\frac{p-1}{2}} \equiv \pm 1 \pmod{p}.$$

Euler's criterion Given that p is not a factor of a, the conguence $x^2 \equiv a \pmod{p}$ has two solutions or no solutions according as:

$$a^{\frac{p-1}{2}} \equiv 1 \text{ or } -1 \pmod{p}.$$

We call a a *quadratic residue* or a *quadratic non-residue* accordingly.

7. By checking the Euler criterion, show decide whether or not 7 is a quadratic residue modulo 31.

For questions 8 and 9 show that the equations in question have solutions and find them.

8. $x^2 \equiv 7 \pmod{31}$.

9. $x^2 \equiv 41 \pmod{61}$.

10. Find conditions on r that will ensure that if a is a quadratic residue $\pmod{p}$ and $ab \equiv r \pmod{p}$, then b is a quadratic residue $\pmod{p}$.

Problem Set 5 Quadratic reciprocity and the Legendre symbol

The *Legendre symbol* $(a/p) = \pm 1$ according as a is or is not a quadratic residue modulo p. The *Quadratic Reciprocity Theorem* says that if $p \equiv q \equiv 3 \pmod 4$, then $(p/q) = -(q/p)$; otherwise $(p/q) = (q/p)$.

For Questions 1-3 show that the Legendre symbol has the following properties.

1. If $a \equiv b \pmod p$ then $(a/p) = (b/p)$.

2. If p is not a factor of a then $(a^2/p) = 1$.

3. If p is not a factor of either a or b then $(ab/p) = (a/p)(b/p)$.

4. Find the values of $(19/5)$ and $(-9/13)$.

5. Find whether or not $x^2 \equiv 85 \pmod{97}$ has a solution.

6. Prove that $(-1/p) = 1$ if $p \equiv 1 \pmod 4$ and $(-1/p) = -1$ otherwise.

7. Given that $(2/p) = 1$ if and only if $p \equiv 1$ or $p \equiv 7$ (modulo 8), find $(3201/8191)$.

8. First show that $x^2 \equiv 14 \pmod{31}$ has solutions and then find them.

9. Show using the Quadratic Reciprocity Theorem to show that if $p = q + 4a$ then $(a/p) = (a/q)$.

10. Find all the solutions of $x^2 \equiv 211 \pmod{159}$.

Problem Set 6 Countable and Uncountable Sets

We call an infinite set S *countable* if there is a *bijection* $f : \mathbb{N} \to S$ (ie, a one-to-one and onto mapping). Finite sets are also considered to be countable. *Cantor's diagonal argument* shows that the real interval $(0,1)$ is *uncountable*, meaning not countable. The *Schroeder-Bernstein* argument shows that if there are *injective* mappings (ie one-to-one functions) from a set A to a set B and from B to A, then there is a bijection between the sets, in which case we say that A and B have the same *cardinal.* A subset of a countable set is easily seen to be either countable or finite.

1. Show that $S = \{x \in \mathbb{R} : 0 < x < \frac{1}{2}\}$ is uncountable.
2. Show that the union

$$A = \bigcup_{n=1}^{\infty} A_n$$

of a list $A_1, A_2, \cdots,$ of countable sets is countable.

3. Prove that the sets $\mathbb{Z}$, of all integers and $\mathbb{Q}$, the set of rationals, are countable sets.

4. Deduce using Question 4 that the set I of all *irrational numbers* is uncountable.

5. Show that the *direct product* (or *Cartesian product*)

$$P = A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \cdots, a_n) : \, a_i \in A_i, \, 1 \leq i \leq n\}$$

of finitely many countable sets $A_i \, (1 \leq i \leq n)$ is itself countable.

6. Show that the result of Question 5 does not extend to countably many factors by putting $A_i = \{0,1\}$ $(i = 1, 2, \cdots)$ and considering $A_1 \times A_2 \times \cdots \times A_n \times \cdots$.

7. Show that the range of a function $f : A \to B$, where A is a countable set, is itself countable.

8. Let A and C be countable sets. If B is some arbitrary set, can you decide whether or not $A \cup (B \cap C)$ is countable?

9. Prove that the set $\mathbb{C}$ of all complex numbers is uncountable.

10. Show that the set of all *algebraic* numbers, which are real numbers that are roots of some polynomial equation with rational coefficients, is countable. Hence deduce that the set of *transcendental numbers* (real numbers that are not algebraic) is uncountable.

Problem Set 7 Difference Equations

1. Solve the *difference equation* $u_n = 2u_{n-1} + 1$, $n = 1, 2, \cdots$ with $u_0 = 0$.

2. The Fibonacci numbers f_n are defined by $f_0 = 0$, $f_1 = 1$ and $f_{n+1} = f_n + f_{n-1}$ for all $n \geq 1$. By substituting $f_n = Aw^n$, find a formula for f_n of the form $f_n = A_1 w_1^n + A_2 w_2^n$.

3. Hence find $\lim_{n\to\infty} \frac{f_{n+1}}{f_n}$.

4. For the second order homogenous linear difference equation

$$u_n = pu_{n+1} + qu_{n-1},$$

with $p+q = 1$, $p \neq q$ by substituting $u_n = Aw^n$ find two candidates $u_n = A_1 w_1^n$ and $u_n = A_2 w_2^n$ for solutions.

5. Check that

$$u_n = A_1 w_1^n + A_2 w_2^n$$

does indeed solve the equation of Question 4.

6. Determine values of A_1 and A_2 so that $u_0 = 0$ and $u_l = 1$ and so find the solution to the equation with these values.

7. The case where $p = q = \frac{1}{2}$ is that of twin roots of the associated quadratic for then we find $w_1 = w_2 = 1 = w$. In that case we seek solutions in the form $u_n = (A_1 + A_2 n)w^n$. Use this approach to solve

$$u_n = \frac{u_{n-1} + u_{n+1}}{2},\ u_0 = 0,\ u_l = 1.$$

8. Solve the inhomogenous difference equation

$$v_n = 1 + pv_{n+1} + qv_{n-1}$$

with $v_0 = v_l = 1$ and $p+q = 1$ by augmenting the solution of the corresponding homogeneous equation by adding the term $f(n) = kn$.

9. Solve the equation of Question 8 for the case $p = q = \frac{1}{2}$ by adding the augmented solution $f(n) = kn^2$ to that of the solution of the corresponding homogeous equation.

10. Hence solve the equation of Question 8 and 9 subject to the initial conditions that $u_0 = 0$ and $u_l = 1$.

Problem Set 8 Generating functions

Suppose that a_r is the number of ways of selecting r objects subject to some constraints. The *generating function* for the a_r is then $g(x) = \sum_{r=0}^{\infty} a_r x^r$, while the *expontential generating function* is $g(x) = \sum_{r=0}^{\infty} \frac{a_r x^r}{r!}$. One particular example used in several of these quesionts is

$$\sum_{r=0}^{\infty} \binom{r+n-1}{r} x^r = \frac{1}{(1-x)^n}.$$

.

1. Find the coefficient of x^r in $(x^2 + x^3 + \cdots)^5$.
2. Find the coefficient of x^{12} in $x^2(1-x)^{-10}$.

3. Find the number of ways of selecting 10 balls from a large pile of red, white, and blue balls if there must be an even number of blue balls.

4. How many ways are there to place an order for 12 drinks if there are 5 types of drinks and at most four drinks of any one type are allowed?

5. Write down the generating function for collecting r euros from 20 people, each of which can give either 1 euro or nothing and the other can give either $0, 1$, or 5 euros. Hence find the number of ways of collecting 15 euros from this group.

6. Find the number of ways to distribute 25 balls into seven distinct boxes if the first box can have no more than 10 balls but the others can hold any amount.

7. How many ways are there to select 25 toys from seven types with between two and six of each type.

8. How many ways are there of getting a sum of 25 when 10 dice are rolled?

9. Use exponential generating functions to find the number of ways of placing 25 people into three rooms with at least one person in each room.

10. Find the number of r-digit *quaternary* sequences (sequences made from the digits $0, 1, 2, 3$) with an even number of 0's and an odd number of 1's.

Problem Set 9 Inclusion-exclusion Principle

Inclusion-exclusion principle Let $\mathcal{U}$ denote a universal set of N elements and let $A_i \subseteq \mathcal{U}$ for $1 \leq i \leq n$. Let S_k denote the sum of the sizes of all the k-tuple intersections of the A_i's . Surpressing the intersection signs we write:

$$|\bar{A}_1\bar{A}_2\cdots\bar{A}_n| = N - S_1 + S_2 - S_3 + \cdots + (-1)^k S_k + \cdots + (-1)^n S_n.$$

1. How many ways are there to select a 5-card hand from a regular 52-card deck such that the hand contains at least one card in each suit?

2. How many ways are there to roll 10 dice with all 6 faces appearing?

3. How many n digit decimal sequences (using $0, 1, 2, \cdots, 9$) are there in which the digits $1, 2$ and 3 all appear?

4. How many ways are there to distribute r distinct objects into five distinct boxes with at least one empty box?

5. Use generating functions to find the number of different integer solutions to the equation $x_1 + x_2 + \cdots + x_6 = 20,\ 0 \leq x_i \leq 8$.

6. Solve Question 5 using inclusion-exclusion.

7. If n leads are plugged randomly into n sockets, what is the probability that no lead is in its correct socket?

8. A permutation α on $\bar{n} = \{1, 2, \cdots, n\}$ for which $i\alpha \neq i$ for all $1 \leq i \leq n$ is call a *derangement*. Find the limiting proportion of derangements of all the permutations on $\bar{n}$.

9. Show that

$$D_n = (n-1)(D_{n-1} + D_{n-2})\ (n \geq 3).$$

10. Use Question 9 to show that

$$D_n = nD_{n-1} + (-1)^n\ (n \geq 2).$$

Problem Set 10 Catalan numbers

The nth *Catalan number* C_n is the number of ways of splitting a regular $(n+2)$-gon into n triangles by non-intersecting diagonals of the polygon.

1. Conventionally we put $C_0 = 1$. Find C_n for $n = 1, 2, 3$.
2. Use an inductive argument to prove the recurrence:

$$C_n = \sum_{k=1}^{n} C_{k-1} C_{n-k}.$$

3. By considering a box of n red and m blue balls, justify the identity:

$$\sum_{k=0}^{n} \binom{n}{k}\binom{m}{k} = \binom{n+m}{m} \text{ for } n \leq m.$$

4. Show that the central binomial coefficient $\binom{2n}{n}$ satisfies:

$$\binom{2n}{n} = \frac{(-\frac{1}{2})(-\frac{3}{2})\cdots(-\frac{2n-1}{2})}{n!}(-4)^n.$$

5. Hence show that the generating function $g(x)$ for the central binomial coefficients is

$$g(x) = \sum_{n=0}^{\infty} \binom{2n}{n} x^n = \frac{1}{\sqrt{1-4x}}, \text{ where } |x| < \tfrac{1}{4}.$$

6. By use of the factorization

$$(1-4x)^{-1} = (1-4x)^{-\frac{1}{2}} \cdot (1-4x)^{-\frac{1}{2}}$$

verify the identity

$$\sum_{k=0}^{n} \binom{2k}{k}\binom{2n-2k}{n-k} = 4^n.$$

7. By replacing x by x^2 and integrating both sides of the equality of Question 5 between 0 and $\frac{1}{4}$ deduce the identity:

$$\sum_{n=0}^{\infty} \frac{1}{4^{2n}(2n+1)}\binom{2n}{n} = \frac{\pi}{3}.$$

8. Let $h(x) = \sum_{k=0}^{\infty} C_k x^k$. Show that

$$(h(x))^2 = \sum_{k=0}^{\infty} C_{k+1} x^k$$

9. Use Question 8 to set up a quadratic equation for $h(x)$ and conclude that

$$h(x) = \frac{1 - \sqrt{1 - 4x}}{2x}$$

10. By expanding $h(x)$ by the Binomial theorem, deduce that

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

Hints for Problems

Problem Set 1

Congruence problems: for $d = \gcd(a, n)$ the equation $ax \equiv b \pmod{n}$ is equivalent to $a'x \equiv b' \pmod{n'}$ where $a' = \frac{a}{d}$, $b' = \frac{b}{d}$ and $n' = \frac{n}{d}$. Adding multiple of the modulus to the RHS then eventually allows cancellation; alternatively acting the Euclidean algorithm on the pair a', n' can be used to find all solutions.

9. Express as three linear congruences. Find the general solution of the first and substitute into the second, solving for the free variable, and continue.

Problem Set 2

One method to solve linear Diophantine equations is to work module a where a is one of the coefficients of x or y (easiest to take the smallest coefficient). This will give one variable in terms of a parameter t that can then be substituted into the equation to find both x and y in terms of t. The solution set can then be formulated, taking into account any further restrictions on the solution type required.

Problem Set 3

No particular hints: just be mindful of the given definitions and defining properties of the parameters that are used to prove various properties listed, such as that in Question 6.

Problem Set 4

1. Use the fact that $ax \equiv 1 \pmod{p}$ has a unique solution to make the coefficient of x^2 equal to 1; however you need to deal separately with the cases where the coefficient of the linear term is even or odd.

4. If r is a solution, then so is $p - r$. Then you need to check that every solution equals either r or $p - r$. Use the difference of two squares and the Euclid

lemma; if a prime divides a product then the prime divides one of the factors of that product.

8. & 9. Add on mulitples of the modulus to the right hand side; upon reaching a number with a square factor s^2, divide both sides by s^2 and continue the calculation with $\left(\frac{x}{a}\right)^2$.

10. Use the Euler criterion.

Problem Set 5

2. Just solve the corresponding equation.

3. Begin with $(a/p) \equiv a^{\frac{p-1}{2}} \pmod{p}$.

6. Again use the Euler criterion.

9. Relate (p/q) and (q/p) to (a/q) and (a/p) respectively, use the CRT and the fact that $p \equiv q \pmod{4}$.

10. Factorize the modulus into the product of two primes and solve in the manner of the Chinese Remainder Theorem substitution technique.

Problem Set 6

1. Argue by contradiction: if $(0, \frac{1}{2})$ were countable, show there would be a bijection $g : \mathbb{N} \to (0, 1)$, which you know does not exist.

2. A bijection from $\mathbb{N}$ onto the set A_i can be regarded as a list of all its members, $a_{i,1}, a_{i,2}, \cdots$. Use all these lists to somehow create one big list for their union.

3. One way to show that $\mathbb{Q}$ is countable is to list its members by breaking $\mathbb{Q}$ into a union of finite sets and combining the lists of these finite subsets into one grand list.

5. By induction, only the $n = 2$ case need be done and again the challenge is create one big list of $A_1 \times A_2$ from given lists for A_1 and A_2.

6. Think binary and that will allows you to set up a bijection from the infinite product set and all the reals in the unit interval, which you know form an uncountable set.

10. Show that the set of roots of polynomials with rational coeffients that have degree at most n is countable. The set A of algebraic numbers is just the union of all such sets and, being then a countable union of countable sets, it will follow that A itself is countable.

Problem Set 7

1. Prove by induction a guess that can easily be made by computing the first few members of the sequence.

Problem Set 8

1. Sume the geometric series.

3 - 8. Expand the product of the sums for each term and go after the relevant coefficients.

9. The exponential generating function in this case is that of $(e^x - 1)^3$; we subtract 1 as we want none of the 3 rooms empty. Writing the coefficients in the form $\frac{a_r}{r!}$ here is relevant as we are interested in the values of a_r as we are not interested in the order in which the distinct people are assigned to a room, hence we divide the overall coefficient by $r!$. This is why the exponential generating function is relevant.

10. Again you want the coefficient of $\frac{x^r}{r!}$ and the exponential generating function works out to be $\frac{1}{3}(e^{4x} - 1)$ as the 'even' and 'odd' restrictions lead to series for cosh and sinh.

Problem Set 9

1. Let each A_i denote the number of hands with a void in each of the four suits. You want $|\bar{A}_1\bar{A}_2\bar{A}_3\bar{A}_4|$.

4. Use the formula

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = S_1 - S_2 + S_3 - \cdots + (-1)^n S_n.$$

5. You want the coefficient of x^{20} in

$$g(x) = (1 + x + x^2 + \cdots + x^8)^6.$$

6. Let A_i be the subset of integer solutions in which $x_i \geq 9$.

Problem Set 10

2. To set up an induction label the vertices of the $(n+2)$-gon N by the integers $1, 2, \cdots, n$ and fix attention on the edge $E = 1\,2$. In any partition of N by non-intersecting triangles, E is the base of some triangle T_k, where k is

the third vertex of T_k $(3 \leq k \leq n+2)$. The sides $1\,k$ and $2\,k$ split N into an $(n-k+4)$-gon and a $(k-1)$-gon respectively.

5. Use the binomial expansion of $(1-4x)^{-\frac{1}{2}}$.

9. Express $x(h(x))^2$ in terms of $h(x)$ and solve the resulting quadratic equation.

Answers to the Problems

Problem Set 1

1. No solution. 2. 4. 3. $\{2,5\}$. 4. 23. 5. $\{16,35,54\}$. 6. 23. 7. 213. 8. $x=3$, $y=0$ 9. 7. 10. 4.

Problem Set 2

1. $x=1-t$, $y=2t$. 2. $x=16t-1$, $y=2-15t$. 3. $x=3$, $y=2$. 4. $\{(x,y):x=-4-5s,\ y=-5-2s,\ s\geq 0\}$. 5.$\{(25,2,4),(24,4,3),(23,6,2),(22,8,1)\}$. 6. 21. 7. 21. 8. 5. 9. 5. 10. £120.

Problem Set 3

1. $p^{m-1}(p-1)$ 3. $n=33$, $\phi(n)=20$, $(7,20)=1$ 4. $d=3$ 5. $M=30$ 7. $M=6$ 8. 646 9. $d=135$ 10. $M=77$.

Problem Set 4

2. 3 or 4. 3. 4 or 5. 7. 7 is a quadratic residue. 8. 10 or 21. 9. 23 or 38. 10. r is a quadratic residue.

Problem Set 5

4. 1,1.5. Yes. 7. -1. 8. 13, 18. 10. $23,76,83,136$.

Problem Set 6

Problem Set 7

1. $2^n - 1$. 2. $f_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$, $n = 0, 1, 2, \cdots$. 3. $\frac{1+\sqrt{5}}{2}$. 4. $A_1, A_2\left(\frac{q}{p}\right)^n$ 6. $\frac{\left(\frac{q}{p}\right)^n - 1}{\left(\frac{q}{p}\right)^l - 1}$ 7. $\frac{n}{l}$ 8. $A_1 + A_2\left(\frac{q}{p}\right)^n + \frac{n}{q-p}$. 9. $A_1 + A_2 n - n^2$ 10. $n(l-n)$.

Problem Set 8

1$\binom{r-6}{r-10}$. 2. 5005 3. 161 4. 320 5.$\binom{19}{15} + \binom{19}{14} + \binom{19}{10}$. 6.$\binom{31}{25} - \binom{20}{14}$. 7. $\binom{17}{11} - 7\binom{12}{6} + \binom{7}{2}\binom{7}{1}$. 8. $\binom{24}{11} - 10\binom{18}{9} + \binom{10}{2}\binom{14}{5}$. 9. $3^{25} - 3 \cdot 2^{25} + 3$ 10 4^{r-1}.

Problem Set 9

1. $\binom{52}{5} - 4\binom{39}{5} + 6\binom{26}{5} - 4\binom{13}{5}$ 2. $6^{10} - 6 \cdot 5^{10} + 15 \cdot 4^{10} - 30 \cdot 3^{10} + 15 \cdot 2^{10} - 6$ 3. $10^n - 3 \cdot 9^n + 3 \cdot 8^n - 7^n$. 4. $5 \cdot 4^r - 10 \cdot 3^r + 10 \cdot 2^r - 5$ 5 & 6. $\binom{25}{20} - 6\binom{16}{11} + 15\binom{7}{2}$ 7. $n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$ 8. e^{-1}.

Problem Set 10

1. $C_1 = 1$, $C_2 = 3$, $C_3 = 5$. 7 $\frac{\pi}{3}$ 9. $h(x) = \frac{1-\sqrt{1-4x}}{2x}$ 10. $\frac{1}{n+1}\binom{2n}{n}$.