# Mathematics 203 Vector Calculus Solutions

Professor Peter M. Higgins

July 3, 2017

## Solutions and Comments for problems

## Problem Set 1

1. Consider the triangle defined by the vectors **a** and **b** with common tail so that the third side corresponds to $\mathbf{a}-\mathbf{b}$. Applying the Cosine rule we obtain

$$||\mathbf{a}-\mathbf{b}||^2 = ||\mathbf{a}||^2 + ||\mathbf{b}||^2 - 2||\mathbf{a}||\,||\mathbf{b}||\cos\theta$$

$$\Rightarrow (a_1-b_1)^2+(a_2-b_2)^2+(a_3-b_3)^2 = (a_1^2+a_2^2+a_3^2)+(b_1^2+b_2^2+b_3^2)-2||\mathbf{a}||\,||\mathbf{b}||\cos\theta$$

$$\Rightarrow -2\mathbf{a}\bullet\mathbf{b} = -2||\mathbf{a}||\,||\mathbf{b}||\cos\theta$$

$$\Rightarrow \mathbf{a}\bullet\mathbf{b} = ||\mathbf{a}||\,||\mathbf{b}||\cos\theta.$$

2. Put $\mathbf{a}=(x_1,x_2,0)$ and $\mathbf{b}=(y_1,y_2,0)$ and square both sides of the final equation of Question 1. Since $0\le\cos^2\theta\le 1$ we obtain

$$(x_1y_1+x_2y_2)^2 \le (x_1^2+x_2^2)^2(y_1^2+y_2^2)^2.$$

*Comment* This inequality extends to three and to higher dimensions.

3.

$$||\mathbf{a}+\mathbf{b}||^2 = (\mathbf{a}+\mathbf{b})\bullet(\mathbf{a}+\mathbf{b}) = \mathbf{a}\bullet\mathbf{a}+\mathbf{b}\bullet\mathbf{b}+2\mathbf{a}\bullet\mathbf{b}$$

$$\le ||\mathbf{a}||^2+||\mathbf{b}||^2+2||\mathbf{a}||\,||\mathbf{b}|| = (||\mathbf{a}||+||\mathbf{b}||)^2$$

$$\Rightarrow ||\mathbf{a}+\mathbf{b}|| \leqq ||\mathbf{a}||+||\mathbf{b}||.$$

4. This can be verified directly from the definition. However, the determinant form of the cross product allows us to gain the result from the fact that interchanging any two rows (or columns) in a determinant only changes its sign:

$$-(\mathbf{a}\times\mathbf{b}) = -\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ b_1 & b_2 & b_3\\ a_1 & a_2 & a_3\end{vmatrix} = \mathbf{b}\times\mathbf{a}.$$

5.

$$\mathbf{a}\bullet(\mathbf{a}\times\mathbf{b}) = (a_1,a_2,a_3)\bullet(a_2b_3-a_3b_2, a_3b_1-a_1b_3, a_1b_2-a_2b_1)$$

$$= a_1a_2b_3-a_1a_3b_2+a_2a_3b_1-a_2a_1b_3+a_3a_1b_2-a_3a_2b_1 = 0$$

as the terms cancel in pairs.

*Comment* Since it is equally the case that $\mathbf{b}\bullet(\mathbf{a}\times\mathbf{b})=0$, the geometric interpretation of this result is that $\mathbf{a}\times\mathbf{b}$ is orthogonal to the plane defined by the pair of vectors **a** and **b**.

6.

$$||\mathbf{a}\times\mathbf{b}||^2 = (a_2b_3-a_3b_2)^2+(a_3b_1-a_1b_3)^2+(a_1b_2-a_2b_1)^2$$

$$= (a_2^2b_3^2 + a_3^2b_2^2 + a_3^2b_1^2 + a_1^2b_3^2 + a_1^2b_2^2 + a_2^2b_1^2) - 2(a_2b_3a_3b_2 + a_3b_1a_1b_3 + a_1b_2a_2b_1);$$

$$||\mathbf{a}||^2||\mathbf{b}||^2 - (\mathbf{a} \bullet \mathbf{b})^2 = (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2$$
$$= (a_1^2b_1^2 + a_2^2b_2^2 + a_3^2b_3^2) + (a_1^2b_2^2 + a_1^2b_3^2 + a_2^2b_1^2 + a_2^2b_3^2 + a_3^2b_1^2 + a_3^2b_2^2) -$$
$$\Big((a_1^2b_1^2 + a_2^2b_2^2 + a_3^2b_3^2) + 2(a_1b_1a_2b_2 + a_1b_1a_3b_3 + a_2b_2a_3b_3)\Big)$$
$$= (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2 = ||\mathbf{a} \times \mathbf{b}||^2.$$

7. From the identity of Question 6 we have by using Question 1:

$$||\mathbf{a} \times \mathbf{b}||^2 = ||\mathbf{a}||^2||\mathbf{b}||^2 - ||\mathbf{a}||^2||\mathbf{b}||^2 \cos^2\theta$$

$$= ||\mathbf{a}||^2||\mathbf{b}||^2(1 - \cos^2\theta) = ||\mathbf{a}||^2||\mathbf{b}||^2 \sin^2\theta$$

and since lengths are non-negative, by taking square roots we obtain

$$||\mathbf{a} \times \mathbf{b}|| = ||\mathbf{a}||\,||\mathbf{b}|| \sin\theta.$$

In particular $||\mathbf{a} \times \mathbf{b}|| = \mathbf{0}$ if and only if at least one of $\mathbf{a}$ or $\mathbf{b}$ is $\mathbf{0}$, or if $\sin\theta = 0$ in which case the vectors $\mathbf{a}$ and $\mathbf{b}$ point in the same or in opposite directions. These conditions are summarised by the condition of the vectors being parallel: $\mathbf{a} = \lambda\mathbf{b}$ for some $\lambda \in \mathbb{R}$.

8.

$$\mathbf{a}\times(\mathbf{b}+\mathbf{c}) = ((a_2(b_3+c_3)-a_3(b_2+c_2),\ a_3(b_1+c_1)-a_1(b_3+c_3),\ a_1(b_2+c_2)-a_2(b_1+c_1))$$

$$= (a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1) + (a_2c_3 - a_3c_2,\ a_3c_1 - a_1c_3,\ a_1c_2 - a_2c_1)$$
$$= (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c}).$$

9.

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \bullet \mathbf{c})\mathbf{b} - (\mathbf{a} \bullet \mathbf{b})\mathbf{c}$$

We verify equality between the respective $x$ components of each side of the equation. By symmetry the result will hold for the other two components, thus establishing the identity.

$$(\mathbf{a} \times (\mathbf{b} \times \mathbf{c}))_1 = (a_2(b_1c_2 - b_2c_1) - a_3(b_3c_1 - b_1c_3))$$
$$= a_2c_2b_1 + a_3b_1c_3 - a_2b_2c_1 - a_3b_3c_1 \quad (1)$$

on the other hand, the first component of the right hand side is

$$(a_1c_1 + a_2c_2 + a_3c_3)b_1 - (a_1b_1 + a_2b_2 + a_3b_3)c_1$$

$$= a_2c_2b_1 + a_3c_3b_1 - a_2b_2c_1 - a_3b_3c_1 \quad (2)$$

We see that (6) and (7) are the same, as required.

10. From Question 9 it follows that$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = -(\mathbf{c}\times(\mathbf{a} \times \mathbf{b})$ if and only if

$$(\mathbf{a} \bullet \mathbf{c})\mathbf{b} - (\mathbf{a} \bullet \mathbf{b})\mathbf{c} = -(\mathbf{c} \bullet \mathbf{b})\mathbf{a} + (\mathbf{c} \bullet \mathbf{a})\mathbf{b}$$

$$\Leftrightarrow (\mathbf{a} \bullet \mathbf{b})\mathbf{c} = (\mathbf{b} \bullet \mathbf{c})\mathbf{a}.$$

In particular, equality can only occur when **a** and **c** are parallel.

## Problem Set 2

1. Let $\mathbf{v}$ be a unit tangent vector to the level surface $f(x, y, z) = c$. The directional derivative of $f$ at any point on this surface is 0 and $f(x, y, z)$ is equal to the constant $c$. This tells us that $\mathbf{v}\bullet(\nabla f) = 0$ and since $\mathbf{v}\neq\mathbf{0}$ it follows that $\nabla f|_P$ for any point $P$ on the surface is a vector orthogonal to the tangent plane at that point.

2. For any point $P$ the directional derivate in the direction of a unit vector $\mathbf{v}$ is

$$\mathbf{v}\bullet(\nabla f|_P) = |\mathbf{v}| \cdot |(\nabla f|_P) \cos\theta$$

where $\theta$ is the angle between the two vectors in question. At the fixed point $P$ the vector $(\nabla f)|_P$ is fixed too and so this quantity takes on the respective maximum and minimum values of this directional derivative will occur when $\cos\theta = \pm 1$, which is to say when $\mathbf{v}$ is chosen to be in the direction of $(\nabla f)_P$ and opposite to the direction of $(\nabla f)_P$ respectively. Hence $(\nabla f)|_P$ points in the direction of most rapid change of the scalar field $f(x, y, z)$ at the point $P$ in question.

3. $\nabla T(x, y, z) = (-e^{-x}, -2e^{-y}, 4e^{4z})$. The temperature increases most rapidly in the direction of the gradient, which at $(1, 1, 1)$ has value $\nabla T(1, 1, 1) = (-e^{-1}, -2e^{-1}, 4e)$; equivalently the direction of $(1, 2, -4e^2)$.

4. We have $f(\mathbf{x}) = x^2yz + 4xz^2$ so that

$$\nabla(\mathbf{x}) = (2xyz + 4z^2.x^2z, x^2y + 8xz)$$

$$f(1, -2, -1) = (2(1)(-2)(-1)+4(-1)^2, 1^2(-1), 1^2(-2)+8(1)(-1)) = (8, -1, 10).$$

$$\mathbf{v} = (2, -1, 2) \Rightarrow |\mathbf{v}| = \sqrt{2^2 + (-1)^2 + (-2)^2} = \sqrt{4 + 1 + 4} = 3;$$

$$\Rightarrow \hat{\mathbf{v}} = \frac{1}{3}(2, -1, 2).$$

We need

$$\nabla f(1, -2, 1) \bullet \hat{\mathbf{v}} = \frac{1}{3}(8, -1, -10) \bullet (2, -1, 2) = \frac{37}{3}.$$

5. The surface is the zero *contour* of $f(x, y, z) = x^2 + y^2 - z$ so that $\nabla f(x, y, z) = (2x, 2y, -1)$; $\nabla(1, 1, 2) = (2, 2, -1) = \mathbf{u}$. Now $|\mathbf{u}| = \sqrt{2^2 + 2^2 + (-1)^2} = 3$; hence $\mathbf{n} = \frac{1}{3}(2, 2, -1)$ is a required normal vector. There is a second possibility in the opposite vector $-\mathbf{n} = \frac{1}{3}(-2, -2, 1)$.

6. $P(2, -1, 2)$ does indeed lie in $S_1 : x^2 + y^2 + z^2 = 9$ as $2^2 + (-1)^2 + 2^2 = 9$ and also in $S_2 : z = x^2 + y^2 - 3$ as $2^2 + (-1)^2 - 3 = 2$. The angle $\theta$ between the surfaces at $P$ equals the angle between the normals at $P$.

$$\nabla f(\mathbf{x}) = (2x, 2y, 2z),\ \nabla g(\mathbf{x}) = (2x, 2y, -1)$$

$$\mathbf{n}_1 = \nabla f|_P = (4,-2,4),\ \mathbf{n}_2 = \nabla g|_P = (4,-2,-1)$$
$$|\mathbf{n}_1| = \sqrt{16+4+16} = 6,\ \mathbf{n}_2 = \sqrt{16+4+1} = \sqrt{21}$$
$$\cos\theta = \frac{\mathbf{n}_1 \bullet \mathbf{n}_2}{|\mathbf{n}_1|\,|\mathbf{n}_2|} = \frac{(4,-2,4)\bullet(4,-2,1)}{6\sqrt{21}} = \frac{16}{6\sqrt{21}} = \frac{8\sqrt{21}}{63};$$
$$\Rightarrow \theta = \cos^{-1}(\frac{8\sqrt{21}}{63}) = 0\cdot 95^c.$$

7.

$$\nabla f = (\frac{(x^2+y^2)-2x^2}{(x^2+y^2)^2}, \frac{-2x^2}{(x^2+y^2)^2}) = (\frac{y^2-x^2}{(x^2+y^2)^2}, -\frac{2x^2}{(x^2+y^2)^2}).$$

Hence $\nabla f(2,3) = \big(\frac{3^2-2^2}{(2^2+3^2)^2}, -\frac{2(2^2)}{(2^2+3^2)^2}\big) = (\frac{5}{169}, -\frac{8}{169}).$

8. At the point $(x_0,y_0) = (2,3)$ we have $z_0 = \frac{2}{2^2+3^2} = \frac{2}{9}$; $f_x(x_0,y_0) = \frac{5}{169}$, $f_y(x_0,y_0) = -\frac{8}{169}$. The equation of the tangent plane at $(x_0,y_0,z_0)$ is

$$f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) = z - z_0, \text{ which gives:}$$
$$\frac{5}{169}(x-2) - \frac{8}{169}(y-3) = z - \frac{2}{9}.$$

9. This follows at once from the linearity of differentiation:

$$\nabla(\lambda f + \mu g) = \big(\frac{\partial(\lambda f+\mu g)}{\partial x}, \frac{\partial(\lambda f+\mu g)}{\partial y}, \frac{\partial(\lambda f+\mu g)}{\partial z}\big)$$
$$= (\lambda\frac{\partial f}{\partial x} + \mu\frac{\partial g}{\partial x}, \lambda\frac{\partial f}{\partial y} + \mu\frac{\partial g}{\partial y}, \lambda\frac{\partial f}{\partial z} + \mu\frac{\partial g}{\partial z})$$
$$= \lambda(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}) + \mu(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z})$$
$$\lambda(\nabla f) + \mu(\nabla g).$$

10.

$$\nabla(fg) = (\frac{\partial(fg)}{\partial x}, \frac{\partial(fg)}{\partial y}, \frac{\partial(fg)}{\partial z})$$
$$= (\frac{\partial f}{\partial x}g + f\frac{\partial g}{\partial x},\ \frac{\partial f}{\partial x}g + f\frac{\partial g}{\partial y},\ \frac{\partial f}{\partial z}g + f\frac{\partial g}{\partial z})$$
$$= (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z})g + f(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z}) = (\nabla f)g + f(\nabla f).$$

## Problem Set 3

1. From the definition we have

$$\operatorname{div}(\mathbf{F}) = \left(\tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial y}, \tfrac{\partial}{\partial z}\right) \bullet (f_1, f_2, f_3) = \tfrac{\partial f_1}{\partial x} + \tfrac{\partial f_2}{\partial y} + \tfrac{\partial f_3}{\partial z}.$$

2. In determinant form we have curl $\mathbf{F} =$

$$\det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ f_1 & f_2 & f_3 \end{bmatrix}$$

$$= \left(\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3}\right)\mathbf{i} + \left(\frac{\partial f_1}{\partial x_3} - \frac{\partial f}{\partial x_1}\right)\mathbf{j} + \left(\frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}\right)\mathbf{k}. \tag{3}$$

In partciular $\nabla \times \mathbf{F} = \mathbf{0}$ if and only if all of the three terms in (8) is identically 0. These three equalities collectively are equivalent to the statement that $\frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i}$ for all $i \neq j$ and of course if $i = j$ the equality is trivial.

3.

$$\nabla \bullet \mathbf{F} = yz + 0 + 1 = 1 + yz.$$

4.

$$\operatorname{curl}\mathbf{F} = (0 - x)\mathbf{i} + (xy - 0)\mathbf{j} + (z - xz)\mathbf{k} = -x\mathbf{i} + xy\mathbf{j} + z(1 - x)\mathbf{k}.$$

5.

$$\nabla \bullet \mathbf{F} = \frac{1}{x} + xze^{xyz} + \frac{1/x}{1 + (z/x)^2} = \frac{1}{x} + \frac{x}{x^2 + z^2} + xze^{xyz}.$$

6.

$$\operatorname{curl}\mathbf{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k} =$$

$$\left(0 - xye^{xyz}\right)\mathbf{i} + \left(0 + \frac{z}{x^2 + z^2}\right)\mathbf{j} + (yze^{xyz} - 0)\mathbf{k}$$

$$= -xye^{xyz}\mathbf{i} + \frac{z}{x^2 + z^2}\mathbf{j} + yze^{xyz}\mathbf{k}.$$

7.

$$\nabla^2 f = \nabla \bullet \nabla f = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \bullet \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

$$\Rightarrow \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

8.

$$\phi_x = k \cos kx \sin ly e^{\sqrt{k^2+l^2}z} \Rightarrow \phi_{xx} = -k^2\phi$$

$$\phi_y = l \sin kx \cos ly e^{\sqrt{k^2+l^2}z} \Rightarrow \phi_{yy} = -l^2\phi$$

$$\phi_z = \sqrt{k^2+l^2}\phi \Rightarrow \phi_{zz} = (k^2+l^2)\phi;$$
$$\Rightarrow \phi_{xx}+\phi_{yy}+\phi_{zz} = (-k^2-l^2+k^2+l^2)\phi = 0.$$

9.
$$\nabla\times(\nabla f) = \nabla\times(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial f}{\partial z})$$
$$= (\frac{\partial}{\partial y}\frac{\partial f}{\partial z}-\frac{\partial}{\partial z}\frac{\partial f}{\partial y},\frac{\partial}{\partial z}\frac{\partial f}{\partial x}-\frac{\partial}{\partial x}\frac{\partial f}{\partial z},\frac{\partial}{\partial x}\frac{\partial f}{\partial y}-\frac{\partial}{\partial y}\frac{\partial f}{\partial x})$$
$$= (0,0,0) = \mathbf{0}.$$

as $\frac{\partial^2 f}{\partial y\partial z} = \frac{\partial^2 f}{\partial z\partial y}$ and so on, with each pair of partial derivatives cancelling.

*Comment* This says that the curl of the gradient of a smooth scalar field is the zero vector: in general a vector field whose curl is identically zero is called *irrotational.*

10.

$$\nabla\bullet(\nabla\times\mathbf{F}) = (\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z})\bullet(\frac{\partial f_3}{\partial y}-\frac{\partial f_2}{\partial z},\frac{\partial f_1}{\partial z}-\frac{\partial f_1}{\partial z}-\frac{\partial f_3}{\partial x},\frac{\partial f_2}{\partial x}-\frac{\partial f_1}{\partial y})$$

$$= \frac{\partial^2 f_3}{\partial x\partial y}-\frac{\partial^2 f_2}{\partial x\partial z}+\frac{\partial^2 f_1}{\partial y\partial z}-\frac{\partial^2 f_3}{\partial y\partial x}+\frac{\partial^2 f_2}{\partial z\partial x}-\frac{\partial^2 f_1}{\partial z\partial y} = 0$$

as each of the three terms carrying a positive sign is matched by another carrying a negative sign with the order of differentiation reversed. Hence they cancel in pairs to give the (scalar) zero.

*Comment* A vector field such as this one whose divergence is identically zero is called *solenoidal.*

## Problem Set 4

1.
$$\nabla\bullet(\mathbf{f}\times\mathbf{g}) = \nabla\bullet(f_2g_3-f_3g_2, f_3g_1-f_1g_3, f_1g_2-f_2g_1)$$
$$= \frac{\partial(f_2g_3-f_3g_2)}{\partial x}+\frac{\partial(f_3g_1-f_1g_3)}{\partial y}+\frac{\partial(f_1g_2-f_2g_1)}{\partial z}$$
$$= \frac{\partial f_2}{\partial x}g_3+f_2\frac{\partial g_3}{\partial x}-\frac{\partial f_3}{\partial x}g_2-f_3\frac{\partial g_2}{\partial x}$$
$$+\frac{\partial f_3}{\partial y}g_1+f_3\frac{\partial g_1}{\partial y}-\frac{\partial f_1}{\partial y}g_3-f_1\frac{\partial g_3}{\partial y}$$
$$+\frac{\partial f_1}{\partial z}g_2+f_1\frac{\partial g_2}{\partial z}-\frac{\partial f_2}{\partial z}g_1-f_2\frac{\partial g_1}{\partial z} \quad (4)$$

Next we work 'from the other end':

$$\mathbf{g}\bullet\nabla \times \mathbf{f} - \mathbf{f} \bullet \nabla\times\mathbf{g}$$

$$= \mathbf{g}\bullet(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}, \frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}, \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y})$$

$$-\mathbf{f}\bullet(\frac{\partial g_3}{\partial y} - \frac{\partial g_2}{\partial z}, \frac{\partial g_1}{\partial z} - \frac{\partial g_3}{\partial x}, \frac{\partial g_2}{\partial x} - \frac{\partial g_1}{\partial y})$$

$$= g_1(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}) + g_2(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}) + g_3(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y})$$

$$-f_1(\frac{\partial g_3}{\partial y} - \frac{\partial g_2}{\partial z}) - f_2(\frac{\partial g_1}{\partial z} - \frac{\partial g_3}{\partial x}) - f_3(\frac{\partial g_2}{\partial x} - \frac{\partial g_1}{\partial y}) \tag{5}$$

and the twelve terms in each of (9) and (10) may be matched, thereby establishing the identity.

2. Applying the identity of Question 1 with $\mathbf{f} = \nabla f$ and $\mathbf{g} = \nabla g$ we obtain

$$\nabla \bullet (\nabla f \times \nabla g) = \nabla g \bullet (\nabla \times \nabla f) - \nabla f \bullet (\nabla \times \nabla g). \tag{6}$$

However, by Question 9 of Set 3 we have $\nabla \times \nabla f = \nabla \times \nabla g = \mathbf{0}$ so the RHS of (11) becomes $\mathbf{0} - \mathbf{0} = \mathbf{0}$.

3. On the LHS we have

$$\nabla \times (\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}, \frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}, \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y})$$

the first component of which is

$$\frac{\partial}{\partial y}(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}) - \frac{\partial}{\partial z}(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x})$$

$$= \frac{\partial^2 v_2}{\partial y \partial x} - \frac{\partial^2 v_1}{\partial y^2} - \frac{\partial^2 v_1}{\partial z^2} + \frac{\partial^2 v_3}{\partial z \partial x}. \tag{7}$$

On the RHS we have

$$\nabla(\nabla \bullet \mathbf{v}) - \nabla^2 \mathbf{v} = \nabla(\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}) - \nabla^2 \mathbf{v}$$

which has first component:

$$\frac{\partial^2 v_1}{\partial x^2} + \frac{\partial v_2}{\partial x \partial y} + \frac{\partial^2 v_3}{\partial x \partial z}$$

$$-\frac{\partial^2 v_1}{\partial x^2} - \frac{\partial^2 v_1}{\partial y^2} - \frac{\partial^2 v_1}{\partial z^2}; \tag{8}$$

we note that the first and fourth terms of (13) cancel leaving the same four terms that we see in (12), as required to complete the verification of equality of the first term. The second and third terms follow the identical pattern up to the naming of the subscripts.

4. Now $\int_C \mathbf{F} \bullet \mathbf{dr} = \int_0^1 \mathbf{F} \bullet \frac{\mathbf{dr}}{\mathbf{dt}} dt =$

$$\int_0^1 ((t^2-t^2),\, (9t^3-t^2),\, (t-t^6))\bullet(1,\, 2t,\, 3t^2)\, dt = \int_0^1 (2t(9t^3-t^2)+3t^2(t-t^6))\, dt =$$

$$\int_0^1 (18t^4 - 2t^3 + 3t^3 - 3t^8)\, dt = \int_0^1 (t^3 + 18t^4 - 3t^8)\, dt = [\frac{t^4}{4} + \frac{18t^5}{5} - \frac{t^9}{3}]_0^1 =$$

$$\frac{1}{4} + \frac{18}{5} - \frac{1}{3} = \frac{15 + 216 - 20}{60} = \frac{211}{60} = 3\frac{31}{60}.$$

5.

$$\mathbf{F}(x, y) = (x^2 + y^2, 2xy + 1) = (\phi_x, \phi_y)$$

$$\Rightarrow \phi = \int (x^2 + y^2)\, dx = \frac{1}{3}x^3 + xy^2 + f(y)$$

$$\Rightarrow \phi_y = 2xy + \frac{df}{dy} = 2xy + 1 \Rightarrow f(y) = y + c.$$

$$\therefore \phi(x, y) = \frac{1}{3}x^3 + xy^2 + y + c.$$

*Comment* We can check: $\phi_x = x^2 + y^2$, $\phi_y = 2xy + 1$. The constant $c$ could be chosen to satisfy a given initial condition for $\phi$.

6.

$$\mathbf{u}(x, y, z) = (x + 2y + 4z, 2x - 3y - z, 4x - y + 2z) = (\phi_x, \phi_y, \phi_z).$$

Hence

$$\phi(x, y, z) = \int (x + 2y + 4z)\, dx = \frac{1}{2}x^2 + 2yx + 4zx + f(y, z)$$

$$\Rightarrow \phi_y = 2x + f(y, z) = 2x - 3y - z \Rightarrow f_y(y, z) = -3y - z$$

$$\Rightarrow f(y, z) = -\frac{3}{2}y^2 - zy + g(z);$$

$$\Rightarrow \phi_z = 4x - y + \frac{dg}{dz} = 4x - y + 2z \Rightarrow g(z) = z^2 + c.$$

$$\Rightarrow \phi(x, y, z) = \frac{1}{2}x^2 + 2xy + 4xz - \frac{3}{2}y^2 - zy + z^2 + c.$$

7.

$$\mathbf{u}(x, y, z) = (xyz, x, z)$$

$$\text{curl}(\mathbf{u}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & x & z \end{vmatrix} = (0 - 0)\mathbf{i} + (0 - xy)\mathbf{j} + (1 - xz)\mathbf{k}$$

$$\text{curl}(\mathbf{u}) = (0, -xy, 1 - xz) \neq \mathbf{0}$$

and hence $\mathbf{u}$ is not conservative for if $\mathbf{u} = \nabla\phi$ say then $\nabla \times \nabla\phi = \mathbf{0}$ by Question 9 Set 3.

8. Let $C$ have a parametrization $\mathbf{r}(t) = (x(t), y(t), z(t))$ $(a \le t \le b)$ so that our integral $I$ takes the form

$$I = \int_a^b \mathbf{f}(\mathbf{r}(t)) \bullet \mathbf{r}'(t)\, dt$$

$$= \int_a^b (\frac{\partial\phi}{\partial x}\frac{dx}{dt} + \frac{\partial\phi}{\partial y}\frac{dy}{dt} + \frac{\partial\phi}{\partial z}\frac{dz}{dt})\, dt$$

$$= \int_a^b \phi'(t)\, dt = \phi(b) - \phi(a).$$

In particular, if $C$ is a closed curve we may take $\mathbf{b} = \mathbf{a}$ and so we get

$$\oint_C \mathbf{f}\, ds = \phi(a) - \phi(a) = 0.$$

9. Since $\mathbf{F}(x, y) = \nabla\phi(x, y)$ is conservative, (where $\phi(x, y) = \frac{1}{3}x^3 + xy^2 + y$) the value of our integral $I$ is independent of the path taken between endpoints and has value

$$I = \phi(\mathbf{b}) - \phi(\mathbf{a}) = \phi(1, 2) - \phi(0, 0).$$

$$\therefore I = (\frac{1}{3}(1^3) + (1)(2^2) + 2 - (0 + 0 + 0)) = 6\frac{1}{3}.$$

10. The line $L$ from $(0, 0)$ to $(1, 2)$ has a parametrization $\mathbf{r}(t) = (t, 2t)$ $(0 \le t \le 1)$, and so $\mathbf{r}'(t) = (1, 2)$. Our vector field $\mathbf{F}(x, y) = (x^2 + y^2, 2xy + 1)$ so that

$$\mathbf{F}(t) = (t^2 + (2t)^2, 2t(2t) + 1) = (5t^2, 4t^2 + 1)$$

$$I = \int_L \mathbf{F}(t) \bullet \mathbf{r}'(t)\, dt = \int_0^1 (5t^2, 4t^2 + 1) \bullet (1, 2)\, dt$$

$$= \int_0^1 (5t^2 + 8t^2 + 2)\, dt = \int_0^1 (13t^2 + 2)\, dt$$

$$= [\frac{13}{3}t^3 + 2t]_0^1 = (\frac{13}{3} + 2) - (0 + 0) = \frac{19}{3} = 6\frac{1}{3}.$$

## Problem Set 5

1. Solve

$$\frac{\partial\phi}{\partial x} = x^2 + y^2 \Rightarrow \phi(x, y) = \frac{x^3}{3} + xy^2 + f(y)$$

$$\Rightarrow \frac{\partial\phi}{\partial y} = 2xy + \frac{dg}{dy} = 2xy + e^{-y} \Rightarrow g(y) = -e^{-y} + c$$

$$\therefore \phi(x,y) = \frac{x^3}{3} + xy^2 - e^{-y} + c.$$

$$I = \phi(0,8) - \phi(1,-1) = (\frac{0^3}{3} + 0(8^2) - e^{-8}) - (\frac{1^3}{3} + (1)(-1)^2 - e^{-(-1)})$$

$$\therefore I = e - \frac{1}{e^8} - \frac{4}{3}.$$

2. Put

$$\frac{\partial\phi}{\partial x} = z\cos x + \ln z \Rightarrow \phi(x,y,z) = z\sin x + x\ln z + f(y,z)$$

$$\Rightarrow \frac{\partial\phi}{\partial y} = \frac{\partial f}{\partial y} = y^2 \Rightarrow f(y,z) = \frac{y^3}{3} + g(z);$$

$$\Rightarrow \phi(x,y,z) = z\sin x + x\ln z + \frac{y^3}{3} + g(z)$$

$$\Rightarrow \frac{\partial\phi}{\partial z} = \sin x + \frac{x}{z} + \frac{dg}{dz} = \sin x + \frac{x}{z} \Rightarrow g(z) = c.$$

Hence the general solution is

$$\phi(x,y,z) = z\sin x + x\ln z + \frac{y^3}{3} + c.$$

Now appying the intitial condition $\phi(\frac{\pi}{2},1,1) = 0$ gives an equation in $c$:

$$\phi(\frac{\pi}{2},1,1) = 1(1) + 1(0) + \frac{1^3}{3} + c = 0 \Rightarrow c = -\frac{4}{3}.$$

$$\therefore \phi(x,y,z) = z\sin x + x\ln z + \frac{y^3}{3} - \frac{4}{3}.$$

3. Put $x(t) = \cos t$, $y(t) = \sin t$, $(0 \leq t \leq 2\pi)$. Then $x'(t) = -\sin t$, $y'(t) = \cos t$ so that our integral $I$ becomes

$$I = \int_0^{2\pi} (2\sin t(-\sin t) - 3\cos t(\cos t))\,dt = -\int_0^{2\pi} (2\sin^2 t + 3\cos^2 t)\,dt$$

$$= -\int_0^{2\pi} (2 + \cos^2 t)\,dt = -2(2\pi) - \frac{1}{2}\int_0^{2\pi} (1 + \cos 2t)\,dt$$

$$= -4\pi - \frac{1}{2}(2\pi) = -5\pi.$$

*Comment* Since $\cos 2t$ has period $\pi$, which divides the length of the interval of integration $(2\pi)$, we know, without further calculation, that its contribution to the value of $I$ is zero.

4.

$$\mathbf{v} = \omega \times \mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix} = \mathbf{i}(\omega_2 z - \omega_3 y) + \mathbf{j}(\omega_3 x - \omega_1 z) + \mathbf{k}(\omega_1 y - \omega_2 x),$$

$$\Rightarrow \text{curl}\mathbf{v} = \big(\nabla \times (\omega \times \mathbf{r})\big) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \omega_2 z - \omega_3 y & \omega_3 x - \omega_1 z & \omega_1 y - \omega_2 x \end{vmatrix}$$

$$= \mathbf{i}(\omega_1 + \omega_1) + \mathbf{j}(\omega_2 + \omega_2) + \mathbf{k}(\omega_3 + \omega_3) = 2\omega$$

$$\therefore \frac{1}{2}\text{curl}\mathbf{v} = \omega.$$

5. The curve of integration consists of four connected line segments, which we parametrize as follows:

$$C_1 : \mathbf{r}_1(t) = (t, 0),\ \mathbf{r}'(t) = (1, 0);$$

$$C_2 : \mathbf{r}_2(t) = (1, t),\ \mathbf{r}_2'(t) = (0, 1);$$

$$C_3 : \mathbf{r}_3(t) = (1 - t, 1),\ \mathbf{r}_3'(t) = (-1, 0)$$

$$C_4 : \mathbf{r}_4(t) = (0, 1 - t), \mathbf{r}_4'(t) = (0, -1);$$

with $0 \le t \le 1$ in all cases. Then our integral $I = \oint_C$ has four parts:

$$\int_{C_1} \mathbf{F} \bullet d\mathbf{r}_1 = \int_0^1 (0, t^2 - t) \bullet (1, 0)\, dt = 0;$$

$$\int_{C_2} \mathbf{F} \bullet d\mathbf{r}_2 = \int_0^1 (t, 1^1 - 1) \bullet (0, 1)\, dt = 0;$$

$$\int_{C_3} \mathbf{F} \bullet d\mathbf{r}_3 = \int_0^1 (1, (1 - t)^2 - (1 - t)) \bullet (-1, 0)\, dt = -\int_0^1 dt = -1;$$

$$\int_{C_4} \mathbf{F} \bullet d\mathbf{r}_4 = \int_0^1 (1 - t, 0^2 - 0)\, dt = 0.$$

$$\therefore \int_C \mathbf{F} \bullet d\mathbf{r} = 0 + 0 - 1 + 0 = -1 \ne 0;$$

and since the *circulation* around this closed curve is not 0, the vector field $\mathbf{F}$ cannot be convervative.

*Comment* If you try to find a potential function $\phi$ for $\mathbf{F}$ in the manner of our previous examples, a contradiction will arise in the that the functions of integration that arise will depend on variables not in their argument, indicating that no solution is possible to the equation $\mathbf{F} = \nabla\phi$.

6. From $\mathbf{F} = m\mathbf{r}''(t)$ we obtain

$$\int_C \mathbf{F} \bullet d\mathbf{r} = m \int_C \mathbf{r}''(t) \bullet \mathbf{r}'(t)\, dt =$$

$$\frac{m}{2}\int_C \frac{d}{dt}\big((r_1'(t)^2 + r_2'(t)^2 + r_3'(t)^2\big)\,dt$$

$$= \frac{m}{2}\int_C \frac{d}{dt}\big(||\mathbf{r}'(t)||^2\big)\,dt = \frac{m}{2}[||\mathbf{r}'(t)||^2]_{\mathbf{r}=A}^{B} = \frac{m}{2}v_B^2 - \frac{m}{2}v_A^2$$

Equating the two expressions for the integral now gives

$$\phi(A) - \phi(B) = \frac{1}{2}mv_B^2 - \frac{1}{2}mv_A^2$$

$$\Rightarrow \phi(A) + \frac{1}{2}mv_A^2 = \phi(B) + \frac{1}{2}mv_B^2.$$

7. We have $\mathbf{r}'(t) = (2t, 2, 3t^2)$ for $0 \leq t \leq 1$. Hence

$$\int_C \mathbf{F}\times d\mathbf{r} = \int_{t=a}^{b} (\mathbf{F}\times\mathbf{r}'(t))\,dt = \int_0^1 (2t^3, -t^3, t^4)\times(2t, 2, 3t^2)\,dt$$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2t^3 & -t^3 & t^4 \\ 2t & 2 & 3t^2 \end{vmatrix} = \mathbf{i}(-3t^5 - 2t^4) - \mathbf{j}(6t^5 - 2t^5) + \mathbf{k}(4t^3 + 2t^4)\text{ so}$$

$$I = \int_0^1 (-3t^5 - 2t^4, 2t^5 - 6t^5, 4t^3 + 2t^4)\,dt = [(-\frac{1}{2}t^6 - \frac{2}{5}t^5, -\frac{2}{3}t^6, t^4 + \frac{2}{5}t^5]_0^1$$

$$= \big(-\frac{9}{10}, -\frac{2}{3}, \frac{7}{5}\big).$$

8. We have $P(x,y) = 2y$ so that $\frac{\partial P}{\partial y} = 2$ and $Q(x,y) = -3x$ so that $\frac{\partial Q}{\partial x} = -3$. Hence, by Green's theorem

$$\int_R (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})\,dxdy = \int_R (-3-2)\,dxdy = -5\pi$$

as $R$ is a unit circle with area $\pi(1^2) = \pi$. This is in accord with our direct calculation of Question 3.

9. To apply Green's theorem we put $P(x,y) = y^3$ and $Q(x,y) = -x^3$ so that $\frac{\partial P}{\partial y} = 3y^2$ and $\frac{\partial Q}{\partial x} = 3x^2$. This gives an integral that is naturally expressed in polar form

$$\oint_C y^3dx - x^3dy = \int\int_R (-3x^2 - 3y^2)\,dxdy = -3\int\int_R r^2\cdot rdrd\theta$$

$$= -3\int_{\theta=0}^{2\pi}\int_{r=0}^{2} r^3\,drd\theta = -6\pi[\frac{r^4}{4}]_0^2 = -6\pi[\frac{16}{4} - 0]$$

$$= -24\pi.$$

10. Here we hae $P(x,y) = \frac{y^3}{3}$ so that $\frac{\partial P}{\partial y} = y^2$ and $Q(x,y) = x - \frac{x^3}{3}$ so that $\frac{\partial Q}{\partial x} = 1 - x^2$. Hence the integrand in Green's theorem is in this instance $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1 - x^2 - y^2 = 1 - r^2$. This quantity is positive throughout the unit

circle and is negative outside it. Hence we maximize the integral by taking $C$ to be the unit circle. This maximum value is then given by

$$\int\int_C (1-r^2)r\,drd\theta = \int_0^{2\pi}\int_0^1 (r-r^3)\,drd\theta = 2\pi[\frac{r^2}{2}-\frac{r^4}{4}]_0^1$$

$$= 2\pi[(\frac{1}{2}-\frac{1}{4})-(0-0)] = \frac{\pi}{2}.$$

## Problem Set 6

1. $P(x,y) = x^2y \Rightarrow \frac{\partial P}{\partial y} = x^2$, $Q(x,y) = 2xy \Rightarrow \frac{\partial Q}{\partial x} = 2y$. Hence

$$\oint_C \mathbf{F}\bullet d\mathbf{r} = \int\int_R (\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y})\,dxdy = \int_{x=0}^1\int_{y=x^2}^x (2y-x^2)dydx$$

$$= \int_0^1 [y^2-x^2y]_{y=x^2}^{y=x}dx = \int_0^1 [(x^2-x^3)-(x^4-x^4)]\,dx$$

$$= \int_0^1 (x^2-x^3)\,dx = [\frac{x^3}{3}-\frac{x^4}{4}]_0^1 = (\frac{1}{3}-\frac{1}{4})-(0-0) = \frac{1}{12}.$$

2. We need to split the boundary curve into two. First the parabolic section $C_1 : y = x^2$ $(0 \le x \le 1)$, so using $x = t$ as parameter we have $\mathbf{r}_1(t) = (t, t^2)$ $(0 \le t \le 1)$ as the two sections meet at $(1,1)$. Hence $\mathbf{r}_1'(t) = (1, 2t)$. The line segment from $(1,1)$ to the origin is $C_2 : \mathbf{r}_2(t) = (1-t, 1-t)$, $(0 \le t \le 1)$ and so $\mathbf{r}_2'(t) = (-1,-1)$. Now $\mathbf{F}(x,y) = (x^2y, 2xy)$ and so $\mathbf{F}(\mathbf{r}_1(t)) = (t^2\cdot t^2, 2t\cdot t^2) = (t^4, 2t^3)$ while

$$\mathbf{F}(\mathbf{r}_2(t)) = ((1-t)^2(1-t), 2(1-t)(1-t) = ((1-t)^3, (1-t)^2).$$

Hence we obtain

$$\oint_C = \int_0^1 (t^4, 2t^3)\bullet(1,2t)\,dt + \int_0^1 ((1-t)^3, 2(1-t)^2)\bullet(-1,-1)\,dt$$

$$= \int_0^1 (t^4+4t^4)\,dt - \int_0^1 ((1-t)^3+2(1-t)^2)\,dt$$

$$= \int_0^1 5t^4\,dt + \int_1^0 u^3+2u^2\,du, \text{ where } u = 1-t$$

$$= [t^5]_0^1 - [\frac{u^4}{4}+\frac{2u^3}{3}]_0^1 = 1-(\frac{1}{4}+\frac{2}{3}) = 1-\frac{3+8}{12} = \frac{1}{12}.$$

3. In the notation of Green's theorem:

$$P(x,y) = x^2y, \Rightarrow \frac{\partial P}{\partial y} = x^2,\ Q(x,y) = -xy^2 \Rightarrow \frac{\partial Q}{\partial x} = -y^2$$

$$I = \int_C \mathbf{F}\bullet d\mathbf{r} = -\int\int_R (x^2+y^2)\,dxdy;$$

converting to polar coordinates we obtain

$$I = -\int_0^{2\pi}\int_0^2 r^2 r\,drd\theta = -2\pi\int_0^2 r^3\,dr = 2\pi[\frac{r^4}{4}]_0^2$$

$$= -2\pi[\frac{16}{4} - 0] = -8\pi.$$

4. Now $C$ has parametrization $\mathbf{r}(t) = (2\cos t, 2\sin t)$ $(0 \le t \le 2\pi)$ so that $\mathbf{r}'(t) = (x'(t), y'(t)) = (-2\sin t, 2\cos t)$. Hence the integral becomes

$$I = \int_0^{2\pi} ((2\cos t)^2(2\sin t)(-2\sin t) - (2\cos t)(2\sin t)^2(2\cos t))\,dt$$

$$= -32\int_0^{2\pi} \cos^2 t\sin^2 t\,dt = -32\int_0^{2\pi}(\frac{1+\cos 2t}{2})(\frac{1-\cos 2t}{2})\,dt$$

$$= -8\int_0^{2\pi}(1-\cos^2 2t)\,dt = -8(2\pi) + 8\int_0^{2\pi}\frac{1+\cos 4t}{2}\,dt$$

$$= -16\pi + 4(2\pi) = -8\pi.$$

5. Take the integral equality in Green's theorem:

$$\oint_C Pdx + Qdy = \int\int_R (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})\,dxdy$$

and put $P(x,y) = -y$ so that $\frac{\partial P}{\partial y} = -1$ and $Q(x,y) = x$ so that $\frac{\partial Q}{\partial x} = 1$. Hence

$$\frac{1}{2}\oint_C xdy - ydx = \frac{1}{2}\int\int_R (1-(-1))dxdy = \int\int_R dxdy = A.$$

6. Using the parametrization $\mathbf{r}(t) = (a\cos t, b\sin t)$ $(0 \le t \le 2\pi)$ we have $\mathbf{r}'(t) = (-a\sin t, b\cos t)$ and so by Question 1

$$A = \frac{1}{2}\int_0^{2\pi}(P,Q)\bullet\mathbf{r}'(t)\,dt = \frac{1}{2}\int_0^{2\pi}(-y,x)\bullet(-a\sin t, b\cos t)\,dt$$

$$= \frac{1}{2}\int_0^{2\pi}(-b\sin t, a\cos t)\bullet(-a\sin t, b\cos t)\,dt$$

$$= \frac{1}{2}\int_0^{2\pi}(ab\sin^2 t + ab\cos^2 t)\,dt = \frac{ab}{2}\int_0^{2\pi}dt = \frac{ab}{2}\cdot 2\pi = \pi ab.$$

7. The standard parametrization of the line segment is given by

$\mathbf{r}(t) = (a,b)+t((c,d)-(a,b)) = (1-t)(a,b)+t(c,d) = ((1-t)a+tc,(1-t)b+td)$, $0 \le t \le 1$. Hence $\mathbf{r}'(t) = (c-a, d-b)$. We thus obtain

$$\int_C xdy - ydx = \int_0^1 x\frac{dy}{dt}dt - \int_0^1 y\frac{dx}{dt}\,dt$$

$$= \int_0^1 ((1-t)a+tc)(d-b)\,dt - \int_0^1 ((1-t)b+td)(c-a)\,dt$$

$$= (d-b)[\frac{t^2}{2}c - \frac{(1-t)^2}{2}a]_0^1 - (c-a)[\frac{t^2}{2}d - \frac{(1-t)^2}{2}b]_0^1$$

$$= (d-b)[(\frac{c}{2}-0)-(0-(\frac{a}{2}))] + (a-c)[(\frac{d}{2}-0)-(0-\frac{b}{2})]$$

$$= \frac{1}{2}(d-b)(c+a) + \frac{1}{2}(a-c)(d+b)$$

$$= \frac{1}{2}(ad+cd-bc-ab+ad+ab-cd-bc) = ad-bc.$$

8. Applying Question 5 and the result of Question 7 to each side of the polygon we obtain the area formula:

$$A = \frac{1}{2}[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + \cdots + (x_{n-1}y_n - x_ny_{n-1})].$$

In the given example we obtain

$$A = \frac{1}{2}[(0\cdot 4-(30)) + (3\cdot 2-(-2)(4)) + ((-2)\dot{(0)} - 2(-1)) + ((-1)(0) - 0\cdot 2)]$$

$$= \frac{1}{2}[0-0+6+8+0+2+0-0] = \frac{16}{2} = 8.$$

9. We parametrize the circle in the usual way: $x(t) = \cos t$, $y(t) = \sin t$ $(0 \le t \le 2\pi)$ so that $x'(t) = -\sin t$, $y'(t) = \cos t$. Hence

$$\oint_C \mathbf{F}\bullet d\mathbf{r} = I = \int_0^{2\pi} (\frac{-\sin t}{\cos^2 t + \sin^2 t}(-\sin t) + \frac{\cos t}{\cos^2 t + \sin^2 t}(\cos t)\,dt$$

$$= \int_0^{2\pi} (\sin^2 t + \cos^2 t)\,dt = \int_0^{2\pi} dt = 2\pi.$$

10.

$$P(x,y) = -\frac{y}{x^2+y^2} \Rightarrow \frac{\partial P}{\partial y} = -\frac{(1)(x^2+y^2)-(y)(2y)}{x^2+y^2} = \frac{y^2-x^2}{x^2+y^2};$$

$$Q(x,y) = \frac{x}{x^2+y^2} \Rightarrow \frac{\partial Q}{\partial x} = \frac{(1)(x^2+y^2)-2x(x)}{x^2+y^2} = \frac{y^2-x^2}{x^2+y^2}.$$

Hence

$$\int\int_R (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})\, dxdy = \int\int_R 0\, dxdy = 0.$$

The answers to Questions 9 and 10 are not equal, showing that Green's theorem does not apply here. However we note that the vector field $\mathbf{F}$ has a singularity within the circle $R$ at the origin, where it is not defined.

## Problem Set 7

1. Parametrize the cone by cylindrical coordinates $\mathbf{r}(t, z) = (z\cos t, z\sin t, z)$ $0 \le t \le 2\pi$, $0 \le z \le 1$. Then

$$\frac{\partial \mathbf{r}}{\partial z} = (\cos t, \sin t, 1),\ \frac{\partial \mathbf{r}}{\partial t} = (-z\sin t, z\cos t, 0)$$

$$\frac{\partial \mathbf{r}}{\partial z} \times \frac{\partial \mathbf{r}}{\partial t} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos t & \sin t & 1 \\ -z\sin t & z\cos t & 0 \end{vmatrix} = \mathbf{i}(-z\cos t) - \mathbf{j}(-z\sin t) + \mathbf{k}(z\cos^2 t + z\sin^2 t)$$

$$\Rightarrow ||\frac{\partial \mathbf{r}}{\partial z} \times \frac{\partial \mathbf{r}}{\partial t}|| = \sqrt{z^2\cos^2 t + z^2\sin^2 t + z^2} = \sqrt{2}z.$$

$$\int_S f(\mathbf{x})\, d\sigma = \sqrt{2}\int_0^{2\pi}\int_0^1 (z\cos t)^2 z\, dzdt$$

$$= \sqrt{2}\int_0^{2\pi} \cos^2 t(\int_0^1 z^3\, dz)\, dt = \sqrt{2}\int_0^{2\pi} \frac{1+\cos 2t}{2}[\frac{z^4}{4}]_0^1\, dt$$

$$= \frac{\sqrt{2}}{8}\int_0^{2\pi} dt = \frac{2\sqrt{2}\pi}{8} = \frac{\sqrt{2}\pi}{4}.$$

2. We use $x$ and $y$ as our paramters for the surface $S$ so that $\mathbf{r}(x, y) = (x, y, g(x, y))$. Hence

$$\frac{\partial \mathbf{r}}{\partial x} = (1, 0, \frac{\partial g}{\partial x}),\ \frac{\partial \mathbf{r}}{\partial y} = (0, 1, \frac{\partial g}{\partial y});$$

$$\Rightarrow \frac{\partial \mathbf{r}}{\partial x} \times \frac{\partial \mathbf{r}}{\partial y} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & \frac{\partial g}{\partial x} \\ 0 & 1 & \frac{\partial g}{\partial y} \end{vmatrix} = \mathbf{i}(-\frac{\partial g}{\partial x}) - \mathbf{j}(\frac{\partial g}{\partial y}) + \mathbf{k};$$

$$\Rightarrow ||\frac{\partial \mathbf{r}}{\partial x} \times \frac{\partial \mathbf{r}}{\partial y}|| = \sqrt{1 + (\frac{\partial g}{\partial x})^2 + (\frac{\partial g}{\partial y})^2}$$

$$\Rightarrow \int\int_S f(x, y, z)\, d\sigma = \int\int_R f(x, y, g(x, y))\sqrt{1 + (\frac{\partial g}{\partial x})^2 + (\frac{\partial g}{\partial y})^2}\, dxdy.$$

3. Let $h(x,y,z) = z - g(x,y)$. Then

$$\nabla h(x,y,z) = (-\frac{\partial g}{\partial x}, -\frac{\partial g}{\partial y}, 1)$$

$$\Rightarrow ||\nabla h(x,y,z)|| = \sqrt{1 + (\frac{\partial g}{\partial x})^2 + (\frac{\partial g}{\partial y})^2}$$

$$\therefore \int\int_S f(x,y,z)\, d\sigma = \int\int_R f(x,y,g(x,y)) ||\nabla h||\, dxdy.$$

4. Here $g(x,y) = \sqrt{x^2+y^2}$ so that

$$\frac{\partial g}{\partial x} = \frac{x}{\sqrt{x^2+y^2}}, \; \frac{\partial g}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$$

$$\Rightarrow ||\nabla h|| = \sqrt{1 + \frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2}} = \sqrt{2}.$$

The region of integration is the projection of the surface onto the $xy$-plane, which in this case is the unit circle. Sticking with cartesian coordinates this gives

$$I = \sqrt{2}\int_{y=-1}^{1}\int_{y=-\sqrt{1-y^2}}^{\sqrt{1-y^2}} x^2 dxdy = \frac{2\sqrt{2}}{3}\int_{-1}^{1}[x^3]_0^{\sqrt{1-y^2}}\, dy$$

$$= \frac{2\sqrt{2}}{3}\int_{-1}^{1}(1-y^2)^{\frac{3}{2}}\, dy.$$

Put $y = \sin t$ so that $dy = \cos t\, dt$;

$$I = \frac{2\sqrt{2}}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(1-\sin^2 t)^{\frac{3}{2}}\cos t\, dt = \frac{2\sqrt{2}}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^4 t\, dt$$

$$= \frac{4\sqrt{2}}{3}\int_0^{\frac{\pi}{2}}(\frac{1+\cos 2t}{2})^2\, dt = \frac{\sqrt{2}}{3}\int_0^{\frac{\pi}{2}}(1 + 2\cos 2t + \cos^2 2t)\, dt$$

$$= \frac{\sqrt{2}}{3}(\frac{\pi}{2} + [\sin 2t]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}}\frac{1+\cos 4t}{2}\, dt)$$

as the period of $\cos 4t$ is $\frac{\pi}{2}$, its contribution to the integral is 0 and we get

$$I = \frac{\sqrt{2}\pi}{6} + 0 + \frac{\sqrt{2}}{6}\frac{\pi}{2} = \frac{\sqrt{2}\pi}{4}.$$

5. Write the surface in the form $y = g(x,z) = 1 - x$ so that $h(x,y,z) = y - g(x,z) = x + y - 1$. Hence

$\nabla h = (1,1,0) \Rightarrow ||\nabla h|| = \sqrt{2},\; f(x,y,z) = x+2y+3z = x+2(1-x)+3z = 2-x+3z$

$$\Rightarrow \int\int_S f(x,y,z)\,d\sigma = \int\int_R f(x,g(x,z),z)||\nabla h||\,dxdz$$
$$= \sqrt{2}\int_{z=0}^{1}\int_{x=0}^{1}(2-x+3z)\,dxdz = \sqrt{2}\int_0^1[2x-\frac{x^2}{2}+3xz]_{x=0}^1\,dz$$
$$= \sqrt{2}\int_0^1(\frac{3}{2}+3z)\,dz = \sqrt{2}[\frac{3}{2}z+\frac{3}{2}z^2]_0^1 = \frac{3\sqrt{2}}{2}(1+1) = 3\sqrt{2}.$$

6. Parametrize the surface $S$ using $x = r\cos t$,$y = r\sin t$ and $z = \frac{h}{R}r$. for $0 \le t \le 2\pi$ and $0 \le r \le R$. Since then $x^2+y^2 = r^2$ and $z$ increases linearly from 0 to $h$ as $r$ increases from 0 to $R$. Hence this gives the surface of an inverted cone. We can find its surface area by integrating the constant function 1 over $S$. Now

$$\frac{\partial \mathbf{r}}{\partial r}\times\frac{\partial \mathbf{r}}{\partial t} = (\cos t, \sin t, \frac{h}{R})\times(-r\sin t, r\cos t, 0)$$
$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos t & \sin t & \frac{h}{R} \\ -r\sin t & r\cos t & 0 \end{vmatrix} = \mathbf{i}(-\frac{hr}{R}\cos t) - \mathbf{j}(-\frac{hr}{R}\sin t) + \mathbf{k}(r\cos^2 t + r\sin^2 t)$$
$$\Rightarrow ||\frac{\partial \mathbf{r}}{\partial r}\times\frac{\partial \mathbf{r}}{\partial t}|| = \sqrt{\frac{h^2r^2}{R^2}\big(\cos^2 t+\sin^2 t\big)+r^2} = \sqrt{\frac{r^2(h^2+R^2)}{R^2}} = \frac{r}{R}\sqrt{h^2+R^2}.$$
$$\Rightarrow A = \int\int_S d\sigma = \int_0^{2\pi}\int_0^R \frac{r}{R}\sqrt{h^2+R^2}\,drdt$$
$$= \frac{\sqrt{h^2+R^2}}{R}\int_0^{2\pi}\big(\int_0^R r\,dr\big)\,dt = \frac{2\pi\sqrt{h^2+R^2}}{R}[\frac{r^2}{2}]_0^R$$
$$= \frac{2\pi\sqrt{h^2+R^2}}{R}\cdot\frac{R^2}{2} = \pi R\sqrt{h^2+R^2}.$$

7.

$$\frac{\partial g}{\partial u}\times\frac{\partial g}{\partial v} = (\cos v, \sin v, 0)\times(-u\sin v, u\cos v, 1) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos v & \sin v & 0 \\ -u\sin v & u\cos v & 1 \end{vmatrix}$$
$$= \mathbf{i}(\sin v) - \mathbf{j}(\cos v) + \mathbf{k}(u\cos^2 v + u\sin^2 v)$$
$$\Rightarrow ||\frac{\partial g}{\partial u}\times\frac{\partial g}{\partial vf}|| = \sqrt{\sin^2 v+\cos^2 v+u^2} = \sqrt{1+u^2}.$$

Hence the required area $A$ of the surface is given by

$$A = \int\int_S d\sigma = \frac{1}{2}\int_0^{3\pi}\int_0^1 \sqrt{1+u^2}\,dudv$$
$$= \int_0^{3\pi}[\frac{1}{2}u(\sqrt{1+u^2}+\frac{1}{2}\ln(u+\sqrt{1+u^2}]_0^1\,dv$$

$$= \frac{1}{2}\int_0^{3\pi} [(\sqrt{2} + \ln(1+\sqrt{2}) - (0+0)]\, dv$$

$$= \frac{3\pi}{2}\big(\sqrt{2} + \ln(1+\sqrt{2})\big).$$

8. We naturally use cylindrical coordinates to parametrize the curved surface of the cylinder: $\mathbf{r}(t, z) = (\cos t, \sin t, z)$ $(0 \le t \le 2\pi)$ $0 \le z \le 1)$. Then

$$\frac{\partial \mathbf{r}}{\partial t} \times \frac{\partial \mathbf{r}}{\partial \mathbf{z}} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin t & \cos t & 0 \\ 0 & 0 & 1 \end{vmatrix} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j};$$

$$\Rightarrow \int_S \mathbf{F} \bullet \mathbf{n}\, \mathbf{d}\sigma = \int_0^{2\pi} \int_0^1 (\cos t, z, -\sin t) \bullet (\cos t, \sin t, 0)\, dzdt$$

$$= \int_0^{2\pi} \int_0^1 (\cos^2 t + z\cos t)\, dzdt = \int_0^{2\pi} [z\cos^2 t + \frac{z^2}{2}\cos t]_{z=0}^1\, dt$$

$$= \int_0^{2\pi} (\cos^2 t + \frac{1}{2}\cos t)\, dt = \frac{1}{2}\int_0^{2\pi} (1 + \cos 2t + \cos t)\, dt$$

$$= \frac{1}{2}[t + \frac{1}{2}\sin 2t + \sin t]_0^{2\pi} = \frac{1}{2}[(2\pi + 0 + 0) - 0] = \pi.$$

9. As in Question 3 we find that

$$\nabla h(x, y, z) = (-\frac{\partial g}{\partial x}, -\frac{\partial g}{\partial y}, 1) = \frac{\partial \mathbf{r}}{\partial x} \times \frac{\partial \mathbf{r}}{\partial y}$$

$$\Rightarrow \int\int_\Sigma \mathbf{F} \bullet \mathbf{n} d\sigma = \int\int_R \mathbf{F} \bullet \frac{\partial \mathbf{r}}{\partial x} \times \frac{\partial \mathbf{r}}{\partial y}\, dxdy = \int\int_R \mathbf{F} \bullet \nabla h\, dxdy.$$

10. We may write the equation of the surface as $z = g(x, y) == 2 - \frac{x}{3} - \frac{y}{2}$ so that

$$h(x, y, z) = z - g(x, y) = \frac{x}{3} + \frac{y}{2} + z - 2$$

$$\Rightarrow \nabla h(x, y, z) = (\frac{1}{3}, \frac{1}{2}, 1);$$

note that this accords with the correct direction of the positve normal to the plane, which is $(2, 3, 6)$. Now

$$\mathbf{F}(x, y, z) = (18z, -12, 3y) = (18(2 - \frac{x}{3} - \frac{y}{2}), -12, 3y)$$

$$= (36 - 6x - 9y, -12, 3y).$$

The surface plane meets the $xy$-plane at the points where

$$2x + 3y = 12 \Rightarrow y = 4 - \frac{2x}{3}; \tag{9}$$

The projection of the surface plane onto the $xy$ plane forms a triangle consisting of the positive axes together with the boundary line given by (14). Hence for a fixed value of $x$ the variable $y$ has limits of 0 and $4-\frac{2x}{3}$. The upper limit for $x$ occurs when $4-\frac{2x}{3}=0$ which gives $x=6$. Hence our integral takes the form:

$$\int\int_S \mathbf{F}(x,y,z)\bullet\mathbf{n}\,d\sigma = \int_0^6\int_0^{4-\frac{2x}{3}}(36-6x-9y,-12,3y)\bullet(\frac{1}{3},\frac{1}{2},1)\,dydx$$

$$\int_0^6\int_0^{4-\frac{2x}{3}}(6-2x)\,dydx = \int_0^6(6-2x)(4-\frac{2x}{3})\,dx$$

$$=\int_0^6(24-4x-8x+\frac{4x^2}{3})\,dx = [24x-6x^2+\frac{4x^3}{9}]_0^6$$

$$=144-216+96-0=24.$$

## Problem Set 8

1. We have that

$$\nabla\bullet\mathbf{F}=\sin y+0-\sin y=0$$

$$\Rightarrow \oiint \mathbf{F}\bullet\mathbf{n}\,d\sigma = \int\int\int_V \nabla\bullet\mathbf{F}\,dxdydz = \int\int\int 0\,dxdydz=0.$$

2. We use the alternative integral $I$ provided by the Divergence theorem:

$$\nabla\bullet\mathbf{F}=2+2y+2z$$

$$\Rightarrow I=\int\int\int_V(2+2y+2z)\,dxdydz.$$

We can argue that, by symmetry, the terms $y$ and $z$ will contribute 0 to $I$ so that $I$ is twice the volume of the sphere, and hence $I=\frac{8}{3}\pi$. We can do the calculation explicitly however by passing to spherical coordinates:

$$J=\int\int\int_V y\,dxdydz = \int_0^{2\pi}\int_0^{\pi}\int_0^1 r\sin\theta\sin\phi\, r^2\sin\phi drd\theta d\phi$$

$$\int\int\int_V y\,dxdydz = \int_0^1\Big(\int_0^{\pi} r\sin^2\phi\Big(\int_0^{2\pi}\sin\theta\,d\theta\Big)d\phi dr\Big)$$

but $\int_0^{2\pi}\sin\phi\,d\phi=0\Rightarrow J=0.$

3.

$$\nabla\bullet\mathbf{F}=y+z+x$$

$$I=\int_0^1\int_0^1\int_0^1(x+y+z)\,dxdydz=\int_0^1\int_0^1[\frac{1}{2}x^2+xy+zx]_0^1dydz$$

$$= \int_0^1 \int_0^1 (\frac{1}{2} + y + z)\, dydz = \int_0^1 [\frac{y}{2} + \frac{y^2}{2} + yz]_0^1\, dz = \int_0^1 (\frac{1}{2} + \frac{1}{2} + z)\, dz$$

$$= \int_0^1 (1 + z)\, dz = [z + \frac{z^2}{2}]_0^1 = 1 + \frac{1}{2} = \frac{3}{2}.$$

4. Here

$$\nabla \bullet \mathbf{F} = 4 - 4y + 2z.$$

In cylindrical coordinates, the volume within the cylinder is represented by

$$\mathbf{r}(r, t, z) = (r \cos t, r \sin t, z),\ \ 0 \le r \le 2,\ 0 \le t \le 2\pi,\ 0 \le z \le 3.$$

By the Divergence theorem, the value of the integral $I$ is

$$I = \int_0^3 \int_0^{2\pi} \int_0^2 (4 - 4r \sin t + 2z) r\, drdtdz$$

$$= \int_0^3 \int_0^{2\pi} \int_0^2 (4r - 4r^2 \sin t + 2rz)\, drdtdz$$

$$= \int_0^3 \int_0^{2\pi} [2r^2 - \frac{4}{3} r^3 \sin t + r^2 z]_{r=0}^2\, dtdz$$

$$= \int_0^3 \int_0^{2\pi} [8 - \frac{32}{3} \sin t + 4z]\, dtdz;$$

the middle term integrates to 0 leaving

$$8\pi \int_0^3 (2 + z)\, dz = 8\pi [2z + \frac{z^2}{2}]_0^3 = 8\pi [6 + \frac{9}{2}]$$

$$= 4\pi \times (12 + 9) = 84\pi.$$

5. To calculate the integral $\int_S \mathbf{F} \bullet \mathbf{n}\, d\sigma$ directly we parametrize the *surface* of the cylinder, the curved portion of which is given by

$$\mathbf{r}(t, z) = (2 \cos t, 2 \sin t, z),\ \ 0 \le t \le 2\pi,\ 0 \le z \le 3.$$

Our next ingredient in our surface integral is

$$\frac{\partial \mathbf{r}}{\partial t} \times \frac{\partial \mathbf{r}}{\partial z} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 \sin t & 2 \cos t & 0 \\ 0 & 0 & 1 \end{vmatrix} = \mathbf{i}(2 \cos t) - \mathbf{j}(-2 \sin t) + \mathbf{k}(0)$$

$= (2 \cos t, 2 \sin t, 0)$, which is indeed outward pointing.

$$\Rightarrow \mathbf{F}(x(t, z), y(t, z), z(t, z)) \bullet \frac{\partial \mathbf{r}}{\partial t} \times \frac{\partial \mathbf{r}}{\partial z} = (8 \cos t, -8 \sin t, z^2) \bullet (2 \cos t, 2 \sin t, 0)$$

$$= 16(\cos^2 t - \sin^3 t).$$

Hence as regards the curved surface of the cylinder our integrand becomes

$$\Rightarrow I = 16\int_0^{2\pi} (\cos^2 t - \sin^3 t)\Big(\int_0^3 dz\Big)\,dt = 48\int_0^{2\pi} (\cos^2 t - \sin^3 t)\,dt.$$

Now

$$48\int_0^{2\pi} \cos^2 t\,dt = 48\int_0^{2\pi} \frac{1+\cos 2t}{2}\,dt = 48(2\pi\cdot\frac{1}{2}+0) = 48\pi.$$

Now $\sin^3 t$ is an odd function which is periodic with period $2\pi$. Hence its integral over any interval of length $2\pi$ is the same and, since $\sin^3 t$ is odd, its integral over the interval $[-\pi, \pi]$ is 0. (Alternatively, substitute $u = \cos t$ and verify this directly.) Hence the contribution to the surface integral of the curved surface of the cylinder is $48\pi$.

The unit outward pointing normals to the top and bottom of the cylinder are respectively $\mathbf{k}$ and $-\mathbf{k}$. The top of the disc is parametrized by $\mathbf{r}(r,t) = (r\cos t, r\sin t, 3)$, $0 \le r \le 2$ and $0 \le t \le 2\pi$. (Note that the $r$ symbol used as a parameter has no particular connection with the generic $\mathbf{r}$ symbol used to indicate a parametrization.) Hence

$$\frac{\partial \mathbf{r}}{\partial r}\times\frac{\partial \mathbf{r}}{\partial t} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos t & \sin t & 0 \\ -r\sin t & r\cos t & 0 \end{vmatrix} = r(\cos^2 t + \sin^2 t)\mathbf{k} = r\mathbf{k}$$

$$\Rightarrow \mathbf{F}(x(r,t), y(r,t), z(r,t)) \bullet \frac{\partial \mathbf{r}}{\partial r}\times\frac{\partial \mathbf{r}}{\partial t} = (4r\cos t, -2r^2\sin^2 t, 9)\bullet(0,0,r) = 9r$$

$$\Rightarrow \oiint_S \mathbf{F}\bullet\mathbf{n}\,d\sigma = \int_0^{2\pi}\int_0^2 9r\,drdt = \frac{9}{2}\int_0^{2\pi} [r^2]_0^2\,dt$$
$$= 18\int_0^{2\pi} dt = 36\pi.$$

For the bottom disc we have $\mathbf{r}(r,t) = (r\cos t, r\sin t, 0)$, $0 \le r \le 2$ and $0 \le t \le 2\pi$. Hence

$$\frac{\partial \mathbf{r}}{\partial r}\times\frac{\partial \mathbf{r}}{\partial t} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos t & \sin t & 0 \\ -r\sin t & r\cos t & 0 \end{vmatrix} = r(\cos^2 t + \sin^2 t)\mathbf{k} = r\mathbf{k}$$

but we must choose the opposite normal $-r\mathbf{k}$ as it is outward pointing. (This direction would come from our calculation if we took the vector product in the opposite order.) Hence

$$\mathbf{F}(x(r,t), y(r,t), z(r,t)) \bullet \frac{\partial \mathbf{r}}{\partial r}\times\frac{\partial \mathbf{r}}{\partial t} = (4r\cos t, -2r^2\sin^2 t, 0)\bullet(0,0,r) = 0$$

$$\Rightarrow \oiint_S \mathbf{F}\bullet\mathbf{n}\,d\sigma = \int_0^{2\pi}\int_0^2 0\,drdt = 0.$$

$$\therefore \oiint_S \mathbf{F} \bullet \mathbf{n}\, d\sigma = 48\pi + 36\pi + 0 = 84\pi,$$

agreeing with the result found in Question 4 via the Divergence theorem.

6. By the Divergence theorem we have

$$\oiint_S \nabla \times \mathbf{F}\, d\sigma = \int\int\int_V \nabla \bullet (\nabla \times \mathbf{F})\, d\sigma = 0,$$

as the divergence of the curl is identically zero (Set 3 Question 10).

7. Now $\nabla \bullet \mathbf{r} = 1 + 1 + 1 = 3$. Hence

$$\oiint_S \mathbf{r} \bullet \mathbf{n}\, d\sigma = \int\int\int \nabla \bullet \mathbf{F}\, dxdydz = 3 \int\int\int_V dxdydz = 3V.$$

8. Applying the Divergence theorem in the reverse direction gives us what we want:

$$\int\int\int_V g\, dxdydz = \int\int\int_V \nabla \bullet \nabla f\, dxdydz = \oiint_S (\nabla f) \bullet n\, d\sigma.$$

9. We parametrize the circle in the usual way: $\mathbf{r}(t) = (\cos t, \sin t)$ for $0 \leq t \leq 2\pi$ and $\mathbf{r}'(t) = (-\sin t, \cos t)$. The (unit) tangent to the curve is $\mathbf{T} = (-\sin t, \cos t)$ and so the outwards normal to this is $\mathbf{n} = (\cos t, \sin t)$ ($-\mathbf{n}$ points inwards). Thus we get

$$\int_C \mathbf{F} \bullet \mathbf{n}\, ds = \int_0^{2\pi} (2\sin t, 5\cos t) \bullet (\cos t, \sin t)\, \sqrt{(-\sin t)^2 + (\cos t)^2} dt$$

$$= \int_0^{2\pi} (2\sin t \cos t + 5\cos t \sin t) \cdot 1\, dt = \frac{7}{2} \int_0^{2\pi} \sin 2t\, dt = 0.$$

Now we evaluate it using the other side of the 2-dimensional Divergence theorem:

$$\nabla \bullet \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} = 0 + 0 = 0$$

$$\Rightarrow \int\int_R \nabla \bullet \mathbf{F}\, dxdy = 0,$$

and so the two calculations agree in this example.

10. In general the unit tangent $\mathbf{T} = (\frac{dx}{ds}, \frac{dy}{ds})$ and the outwards normal is then $\mathbf{n} = (\frac{dy}{ds}, -\frac{dx}{ds})$ (in general, rotating the vector $(a, b)$ through $-\frac{\pi}{2}$ is $(b, -a)$). Hence we may write $\mathbf{n}ds = (dy, -dx)$. Hence we get:

$$\oint_C \mathbf{F} \bullet \mathbf{n}\, ds = \oint_C (P, Q) \bullet (dy, -dx) = \oint_C Pdy - Qdx = \oint_C Pdy - (-Q)dx$$

which by Green's theorem is equal to

$$\int\int_R (\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y})\, dxdy = \int\int_R \nabla \bullet \mathbf{F}\, dxdy.$$

## Problem Set 9

1. We calculate the first ingredient:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 & y^2 & x \end{vmatrix} = \mathbf{i}(0-0) - \mathbf{j}(1-2z) + \mathbf{k}(0-0) = (2z-1)\mathbf{j}.$$

For our surface, we may take any surface with the triangle as boundary, and the obvious candidate is the solid triangle (or *lamina*) itself. Two vectors in the plane are $(1,0,0)-(0,1,0)=(1,-1,0)$ and $(1,0,0)-(0,0,1)=(1,0,-1)$. A normal to the plane is then given by

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{vmatrix} = \mathbf{i}+\mathbf{j}+\mathbf{k}$$

so the plane has equation $x+y+z=c$ and by substitution of any of the points we get that $c=1$. We write this surface in the form $z=g(x,y)=1-x-y$. Our function $h = z - g(x,y) = z-(1-x-y) = x+y+z-1$ and so $\nabla h(x,y,z) = (1,1,1)$. This vector points in the direction consistent with the right hand rule as we traverse the vertices of the triangle in the specified order. Denoting the projected region of the triangle $S$ on to the $xy$-plane by $D$ we see that, the required integral $I$ has the form:

$$I = \int\int_S \nabla \times \mathbf{F}\, d\sigma = \int\int_D (2z-1)\mathbf{j} \bullet (\mathbf{i}+\mathbf{j}+\mathbf{k})\, dxdy = \int\int_D (2z-1)\, dxdy =$$

$$= \int\int_D (2(1-x-y)-1)\, dxdy = \int\int_D (1-2x-2y)\, dxdy.$$

The triangle $D$ is bound by the $x-$ and $y$-axes together with the line $x+y=1$ so that $y=1-x$. Hence we may now calculate:

$$I = \int_0^1 \int_{y=0}^{1-x} (1-2x-2y)\, dydx = \int_0^{1-x} [y-2xy-y^2]_0^{1-x}\, dx$$

$$= \int_0^1 ((1-x)-2x(1-x)-(1-x)^2-(0-0-0))\, dx$$

$$= \int_0^1 (1-x-2x+2x^2-1+2x-x^2)\, dx$$

$$= \int_0^1 (x^2-x)\, dx = [\frac{x^3}{3} - \frac{x^2}{2}]_0^1 = \frac{1}{3} - \frac{1}{2} = -\frac{1}{6}.$$

2. Let $C_1$ be the initial line segment parametrized as $\mathbf{r}_1(t) = (0,t,0)$ $(0 \le t \le 1)$ so that $\mathbf{r}_1'(t) = (0,1,0)$. Then $\mathbf{F}(\mathbf{r}_1(t)) = (t,0,0)$ and so

$$\int_{C_1} \mathbf{F} \bullet d\mathbf{r}_1 = \int_0^1 (t,0,0) \bullet (0,1,0)\, dt = \int_0^1 0\, dt = 0.$$

Next let $C_2$ be the quarter circle, which is parametrized as $\mathbf{r}_2(t) = (0, \sin t, \cos t)$ $0 \le t \le \frac{\pi}{2}$) (Note that begins at $(0,0,1)$ and ends at $(0,1,0)$ as required.) We have $\mathbf{r}_2'(t) = (0, \cos t, -\sin t)$ and $\mathbf{F}(\mathbf{r}_2(t)) = (\sin t, \cos t, 0)$ so that

$$\int_{C_2} \mathbf{F} \bullet d\mathbf{r} = \int_0^{\frac{\pi}{2}} (\sin t, \cos t, 0) \bullet (0, \cos t, -\sin t)\, dt = \int_0^{\frac{\pi}{2}} \cos^2 t\, dt$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} (1 + \cos 2t)\, dt = \frac{\pi}{4} + \frac{1}{4}[\sin 2t]_0^{\frac{\pi}{2}} = \frac{\pi}{4} + \frac{1}{4}[(0-0] = \frac{\pi}{4}.$$

Finally let $C_3$ be the line from $(0,1,0)$ to the origin:$\mathbf{r}_3(t) = (0, 1-t, 0)$ $(0 \le t \le 1)$ so that $\mathbf{r}_3'(t) = (0, -1, 0)$. We have $\mathbf{F}(\mathbf{r}_3(t)) = (-1, 0, 0)$ and so

$$\int_{C_3} \mathbf{F} \bullet d\mathbf{r} = \int_0^1 (-1, 0, 0) \bullet (0, -1, 0)\, dt = \int_0^1 0\, dt = 0.$$

$$\therefore \oint_C \mathbf{F} \bullet d\mathbf{r} = \int_{C_1} \mathbf{F} \bullet d\mathbf{r}_1 + \int_{C_2} \mathbf{F} \bullet d\mathbf{r}_2 + \int_{C_3} \mathbf{F} \bullet d\mathbf{r}_3 = 0 + \frac{\pi}{4} + 0 = \frac{\pi}{4}.$$

3. On the other hand we calculate

$$\nabla \times \mathbf{F}(x,y,z) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = \mathbf{i}(0-1) - \mathbf{j}(1-0) + \mathbf{k}(0-1) = -(\mathbf{i} + \mathbf{j} + \mathbf{k}).$$

As our surface with $C$ as boundary, we take the solid quarter circle with parametrization $\mathbf{r}(r,t) = (0, r\cos t, r\sin t)$ $(0 \le r \le 1, 0 \le t \le \frac{\pi}{2})$. Also $\mathbf{F}(\mathbf{r}(t)) = (r\cos t, r\sin t, 0)$.

$$\frac{\partial \mathbf{r}}{\partial r} \times \frac{\partial \mathbf{r}}{\partial t} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & \cos t & \sin t \\ 0 & -r\sin t & r\cos t \end{vmatrix} = r\mathbf{i}.$$

However, given the orientation of $C$, the right handed normal points in the negative $x$-direction, and since $r \ge 0$ we need to take $-r\mathbf{i}$ instead of $r\mathbf{i}$ (which would have happened if we had taken the parameter $t$ before $r$ and used the reverse vector product). Hence our integral becomes

$$I = \int_S \nabla \times \mathbf{F} \bullet \mathbf{n}\, d\sigma = -\int_{t=0}^{\frac{\pi}{2}} \int_{r=0}^{1} (1,1,1) \bullet (-r, 0, 0)\, dr dt = \int_0^{\frac{\pi}{2}} \int_0^1 r\, dr dt$$

$$= \int_0^{\frac{\pi}{2}} r\, dr dt = \frac{1}{2}\int_0^{\frac{\pi}{2}} [r^2]_0^1\, dt = \frac{1}{2}\frac{\pi}{2} = \frac{\pi}{4}.$$

4.

$$\nabla \times \mathbf{F}(x,y,z) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin x - \frac{y^3}{3} & \cos y + \frac{x^3}{3} & xyz \end{vmatrix}$$

$$= \mathbf{i}(xz - 0) - \mathbf{j}(yz - 0) + \mathbf{k}(x^2 + y^2) = xz\mathbf{i} - yz\mathbf{j} + (x^2 + y^2)\mathbf{k}.$$

The simplest surface to use is $S = \{(x, y, z) : x^2 + y^2 = 1, z = 1\}$. We parametrize $S$ by $\mathbf{r}(r, t) = (r\cos t, r\sin t, 1)$ $(0 \le r \le 1,\ 0 \le t \le 2\pi)$ so that

$$\frac{\partial \mathbf{r}}{\partial r} \times \frac{\partial \mathbf{r}}{\partial t} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos t & \sin t & 0 \\ -r\sin t & r\cos t & 0 \end{vmatrix} = \mathbf{i}(0-0) - \mathbf{j}(0-0) + \mathbf{k}(r\cos^2 t + r\sin^2 t) = r\mathbf{k},$$

which is oriented upwards, which is consistent with a right handed system as applied to $C$. Also $\nabla \times \mathbf{F}(\mathbf{r}(t)) = (r\cos t, r\sin t, r^2\cos^2 t + r^2\sin^2 t) = (r\cos t, r\sin t, r^2)$. Hence our integral $I$ is given by

$$I = \int_0^{2\pi}\int_0^1 (r\cos t, r\sin t, r^2) \bullet (0, 0, r)\, drdt = \int_0^{2\pi}\int_0^1 r^3\, drdt$$

$$= \frac{1}{4}\int_0^{2\pi} [r^4]_0^1\, dt = \frac{1}{4}(2\pi) = \frac{\pi}{2}.$$

5. We are given that $\nabla \times \mathbf{F} = \mathbf{0}$ so that, taking any orientable surface that has $C$ as boundary curve, we have by Stokes's theorem

$$\oint_C \mathbf{F} \bullet d\mathbf{r} = \int\int_S (\nabla \times \mathbf{F}) \bullet \mathbf{n}\, d\sigma = \int\int_S \mathbf{0} \bullet \mathbf{n}\, d\sigma = \int\int_S 0\, d\sigma = 0.$$

*Comment* The *Fundamental theorem of vector fields* says that any smooth vector field can be written as the sum of an irrotational (zero curl) vector field and a solenoidal (zero divergence) vector field.

6. Consider the statement of Stokes's theorem

$$\int\int_S \nabla \times \mathbf{F} \bullet \mathbf{n}\, d\sigma = \oint_C \mathbf{F} \bullet d\mathbf{r}. \tag{10}$$

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F & G & 0 \end{vmatrix} = \mathbf{i}\big(0 - \frac{\partial Q}{\partial z}\big) - \mathbf{j}\big(0 - \frac{\partial P}{\partial z}\big) + \mathbf{k}\big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\big),$$

and since $P$ and $Q$ are independent of $z$ we have $\nabla \times \mathbf{F} = (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})\mathbf{k}$. Also the unit normal $\mathbf{n}$ to our surface $S$ is clearly $\mathbf{k}$, as we can calculate explicitly as for the RHS of (15), take $x$ and $y$ as the parameters of the surfaceand remembering that $z = 0$ we then have $\mathbf{r}(x, y) = (x, y, 0)$. Hence $\frac{\partial \mathbf{r}}{\partial x} = \mathbf{i}$, $\frac{\partial \mathbf{r}}{\partial y} = \mathbf{j}$ so that $\frac{\partial \mathbf{r}}{\partial x} \times \frac{\partial \mathbf{r}}{\partial y} = \mathbf{i} \times \mathbf{j} = \mathbf{k}$. Hence the LHS of (15) beomes:

$$\int\int_S \nabla \times \mathbf{F} \bullet \mathbf{n}\, d\sigma = \int\int_S (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})\mathbf{k} \bullet \mathbf{k}\, dxdy = \int\int_S (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})\, dxdy. \tag{11}$$

On the other hand, for any parametrization $\mathbf{r}(t)$ of $C$ we have

$$\oint_C \mathbf{F} \bullet d\mathbf{r} = \oint_C (P(x, y), Q(x, y), 0) \bullet (\frac{dx}{dt}, \frac{dy}{dt}, 0)\, dt$$

$$= \oint_C P(x,y)\,dx + Q(x,y)\,dy \tag{12}$$

Combining (17) with (16) gives Green's theorem:

$$\oint_C P(x,y)\,dx + Q(x,y)\,dy = \int\int_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy.$$

7. By the Divergence theorem we have

$$\oiint_S \mathbf{E} \bullet \mathbf{n}\,d\sigma = \int\int\int_V \nabla \bullet \mathbf{E}\,dV,$$

which by Maxwell's equation for the divergence of $\mathbf{E}$ is equal to

$$\frac{1}{\varepsilon_0}\int\int\int_V \rho\,dxdydz = \frac{Q}{\varepsilon_0},$$

where $Q$ is the total charge enclosed by $S$.

8. Using Stokes's theorem we obtain

$$\oint_C \mathbf{E} \bullet d\mathbf{r} = \int\int_S \nabla \times \mathbf{E} \bullet \mathbf{n}\,d\sigma$$

which, by Maxwell's equation for the curl of $\mathbf{E}$ gives

$$-\int\int_S \frac{\partial \mathbf{B}}{\partial t} \bullet \mathbf{n}\,d\sigma = -\frac{\partial}{\partial t}\int\int_S \mathbf{B} \bullet \mathbf{n}\,d\sigma.$$

9.

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

$$\Rightarrow \nabla \bullet (\nabla \times \mathbf{B}) = \mu_0 \nabla \bullet \mathbf{J} + \mu_0 \varepsilon_0 \nabla \bullet \frac{\partial \mathbf{E}}{\partial t}$$

and since the divergence of the curl is zero, we can cancel the positive factor $\mu_0$ and take the differential operator outside the integral to obtain

$$\nabla \bullet \mathbf{J} + \varepsilon_0 \frac{\partial (\nabla \bullet \mathbf{E})}{\partial t} = 0$$

and so by Maxwell's equation for the divergence of $\mathbf{E}$,

$$\nabla \bullet \mathbf{J} + \varepsilon_0 \frac{\partial (\frac{\rho}{\varepsilon})}{\partial t} = 0$$

$$\Rightarrow \frac{\partial \rho}{\partial t} + \nabla \bullet \mathbf{J} = 0.$$

10. On the other hand, applying the identity $\nabla \bullet (\mathbf{f} \times \mathbf{g}) = g \bullet (\nabla \times \mathbf{f}) - \mathbf{f} \bullet (\nabla \times \mathbf{g})$ to $\nabla \bullet \mathbf{P}$ we have

$$\nabla \bullet (\mathbf{E} \times \mathbf{B}) = (\nabla \times \mathbf{E}) \bullet \mathbf{B} - (\nabla \times \mathbf{B}) \bullet \mathbf{E}$$

$$= -\frac{\partial \mathbf{B}}{\partial t} \bullet \mathbf{B} - \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \bullet \mathbf{E}$$

$$\Rightarrow -\nabla \bullet (\mathbf{E} \times \mathbf{B}) = \frac{1}{2}\frac{\partial(\mathbf{B} \bullet \mathbf{B})}{\partial t} + \frac{1}{2c^2}\frac{\partial(\mathbf{E} \bullet \mathbf{E})}{\partial t} = \frac{\partial w}{\partial t};$$

$$\therefore \frac{\partial w}{\partial t} + \nabla \bullet \mathbf{P} = 0.$$

## Problem Set 10

1. Write $\mathbf{r}_i(t) = (x_i(t), y_i(t), z_i(t)$ $(i = 1, 2)$. Then, surpressing the symbol $t$ we have $\mathbf{r}_1 \bullet \mathbf{r}_2 = (x_1 x_2, y_1 y_2, z_1 z_2)$ so that

$$(\mathbf{r}_1 \bullet \mathbf{r}_2)' = (x_1' x_2 + x_1 x_2', y_1' y_2 + y_1 y_2', z_1' z_2 + z_1 z_2')$$

$$= (x_1' x_2, y_1' y_2, z_1' z_2) + (x_1 x_2', y_1 y_2', z_1 z_2')$$

$$= \mathbf{r}_1' \bullet \mathbf{r}_2 + \mathbf{r}_1 \bullet \mathbf{r}_2'.$$

2. We have

$$(\mathbf{r}_1 \times \mathbf{r}_2)' = (y_1 z_2 - z_1 y_2,\ z_1 x_2 - x_1 z_2,\ x_1 y_2 - x_2 y_1)'$$

$$= (y_1' z_2 + y_1 z_2' - z_1' y_2 - z_1 y_2',\ z_1' x_2 + z_1 x_2' - x_1' z_2 - x_1 z_2',\ x_1' y_2 + x_1 y_2' - x_2' y_1 - x_2 y_1')$$

$$= (y_1' z_2 - z_1' y_2,\ z_1' x_2 - x_1' z_2,\ x_1' y_2 - x_2' y_1) + (y_1 z_2' - z_1 y_2',\ z_1 x_2' - x_1 z_2',\ x_1 y_2' - x_2 y_1')$$

$$= \mathbf{r}_1' \times \mathbf{r}_2 + \mathbf{r}_1 \times \mathbf{r}_2'.$$

3. Applying the identity of Question 1 we get:

$$(\mathbf{r} \bullet \mathbf{r})' = c' \Rightarrow \mathbf{r}' \bullet \mathbf{r} + \mathbf{r} \bullet \mathbf{r}' = 0$$

$$\Rightarrow 2\mathbf{r} \bullet \mathbf{r}' = 0 \Rightarrow \mathbf{r} \perp \mathbf{r}'.$$

Putting $\mathbf{r} = \beta'(s) = T(s)$, and noting that $\beta'(s) \bullet \beta'(s) = 1$ we therefore conclude that $\beta'(s) \perp \beta''(s)$, which is to say that $\beta''(s) \perp T(s)$.

4. Since $N(s) = \frac{T'(s)}{k(s)}$ we get at once that $T'(s) = k(s)N(s)$, the first Frenet equation.

5. Since $B(s) \bullet B(s) = 1$, it follows from Question 3 that $B'(s) \perp B(s)$. Next note that from $B \bullet T = 0$ we get $(B \bullet T)' = B' \bullet T + B \bullet T' = 0$; however $B \bullet T' = B \bullet kN = k(B \bullet N) = 0$ from the First Frenet-Serret equation and so we also infer that $B' \bullet T = 0$, which is to say that $B'(s) \perp T(s)$. Since $B'$ is perpendicular to both $B$ and $T$ it follows that $B'(s)$ can be written as a multiple of $N(s)$, so we may define $\tau(s)$ by the relation $B'(s) = -\tau(s)N(s)$, thus giving the so-called Third Frenet equation.

6. By the identity of Question 2 we obtain:

$$N' = (B \times T)' = B' \times T + B \times T'$$

and so using Frenet-Serret equations 1 and 3 we get

$$N' = (-\tau N) \times T + B \times kN = (-\tau)(-B) + k(-T) = \tau B - kT.$$

7. The results of the three Frenet equations may be summarised as

$$\begin{pmatrix} T' \\ N' \\ B' \end{pmatrix} = \begin{pmatrix} 0 & k & 0 \\ -k & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

8.

$$T(s) = \beta'(s) = (-\frac{aw}{c}\sin\frac{ws}{c}, \frac{aw}{c}\cos\frac{ws}{c}, \frac{b}{c});$$

putting $||T(s)|| = 1$ gives

$$\frac{a^2w^2}{c^2}(\sin^2\frac{ws}{c} + \cos^2\frac{aw}{c}) + \frac{b^2}{c^2} = 1$$

$$\Rightarrow c^2 = a^2w^2 + b^2.$$

9.

$$T'(s) = (-\frac{aw^2}{c^2}\cos\frac{ws}{c}, -\frac{aw^2}{c^2}\sin\frac{ws}{c}, 0);$$

hence $||T'(s)|| = \frac{aw^2}{c^2} = k(s)$, a constant independent of $s$. Moreover $N(s) = \frac{T'(s)}{k} = -(\cos\frac{ws}{c}, \sin\frac{ws}{c}, 0)$.

10.

$$B = T \times N = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\frac{aw}{c}\sin\frac{ws}{c} & \frac{aw}{c}\cos\frac{ws}{c} & \frac{b}{c} \\ -\cos\frac{ws}{c} & -\sin\frac{ws}{c} & 0 \end{vmatrix}$$

$$= (\frac{b}{c}\sin\frac{ws}{c})\mathbf{i} - (\frac{b}{c}\cos\frac{ws}{c})\mathbf{j} + (\frac{aw}{c}\sin^2\frac{ws}{c} + \frac{aw}{c}\cos^2\frac{ws}{c})\mathbf{k}$$

$$= (\frac{b}{c}\sin\frac{ws}{c}, -\frac{b}{c}\cos\frac{ws}{c}, \frac{aw}{c}).$$

Hence

$$B(s) = (\frac{bw}{c^2}\cos\frac{ws}{c}, \frac{bw}{c^2}\sin\frac{ws}{c}, 0);$$

hence using $B'(s) = -\tau(s)N(s)$ we see that the torsion function is the constant $\tau = \frac{bw}{c^2} = \frac{bw}{a^2w^2+b^2}$.

*Comment* The *Fundamental theorem of space curves* says that curves are characterised by their curvature and torsion in that if two curves share the same curvature and torsion functions then one curve can be mapped onto the other by a *rigid motion*. In particular, any curve with constant curvature and torsion is a helix.