Mathematics 207 Real Analysis

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The purpose of real analysis is to provide a rigorous foundation for the techniques of calculus, which are based on the notion of *limit*. The exercises assume familiarity with the basic ideas of *convergence of a sequence* of real numbers and the definition of *continuity of a function* in terms of the standard symbols $\varepsilon > 0$ and $\delta > 0$ along with the definition of *derivative*. We also assume the *Fundamental theorem of Calculus* and take for granted the *integrability of any continuous function*. The known nature of the *real numbers* is assumed, including the existence of the *greatest lower bound* of a set bounded below and similarly the *least upper bound* of a set bounded above. Set 1 establishes the elementary properties of convergent sequences of real numbers. Set 2 is concerned with certain limits that are especially important, particularly those involving the number e. Set 3 introduces results and examples on continuity of a function. Throughout we will work mainly with one variable mappings although we occasionally expand to matters of several variables. Sets 4 and 5 concern series. We introduce and work with the standard tests for convergence and examples include the *binomial series* for non-integral powers. We draw on all this knowledge in the second part of the module.

In Set 6 we study continuous functions on closed intervals (the prototype of so-called *compact sets*, which we shall meet in Level 3 modules in a more general setting). We prove the *Intermediate value* and *Extreme value* theorems for continuous functions on a closed interval and illustrate the ideas involved with relevant examples. Set 7 introduces the concept of *uniform continuity* for individual and for sequences of functions. This condition is key in justifying many of the techniques of calculus that involve the interchange of limiting operations, such as *term-by-term differentiation and integration* of series. In Set 8 we study *power series* where the uniform convergence of the series within its *radius of convergence* is a crucial property in calculations involving power series representation of functions of interest. In particular the *Weierstrass M-test* is a tool we first meet here. Set 9 introduces and proves another fundamental result of calculus, that being the *Mean value theorem* in various forms and we use the MVT to prove theorems often used in calculus including *Equality of mixed partial derivates*. Set 9 and all of Set 10 are about *Taylor series* and we introduce a study of the *Remainder term* both in the *Lagrange form*, based on the Mean value theorem, and the *Integral form*. We close with some practical calculations including a brief visit into the realm of Taylor series of several variables.

Problem Set 1: Convergence of sequences

Let $(a_n)_{n\geq 1}, (b_n)_{n\geq 1}$, etc. be sequences with limits A, B etc. The symbols λ, μ etc stand for constants. The symbol $=:$ means 'equals by definition'.

1. Prove that if $a_n \leq M$ for all $n \geq 1$ then the same is true of the limit $A = \lim_{n\to\infty} a_n$ (assuming A exists).

2. Prove that $(\lambda a_n + \mu b_n) \to \lambda A + \mu B$.

3. (a) Prove that a convergent sequence $(a_n)_{n\geq 1}$ is *bounded*, meaning that for some $M \geq 0$ we have $|a_n| < M$ for all $n \geq 1$.
(b) Show that a monotonic increasing (resp. decreasing) sequence converges if and only if it is bounded above (resp. below).
(c) Show that if $a_n \to A$ then $|a_n| \to |A|$.
(d) Is the converse to (c) true?

4. Prove that $(a_n b_n)_{n\geq 1} \to AB$.

5. Show similarly that provided $b_n \neq 0$ for all $n \geq 1$ and $B \neq 0$ then $\frac{a_n}{b_n} \to \frac{A}{B}$.

6. Show that any subsequence $(a_{n_i})_{i\geq 1}$ of a convergent sequence $(a_n)_{n\geq 1}$ converges to the same limit.

7. A sequence $(a_n)_{n\geq 1}$ is said to be *Cauchy convergent* if for any $\varepsilon > 0$ there exists N such that for all $m, n \geq N$, $|a_n - a_m| < \varepsilon$. Show that any convergent sequence is Cauchy convergent.

Comment We assume without proof that the converse of the result of Question 7 holds for the real numbers (and similarly for $\mathbb{C}$). We say that the numbers systems $(\mathbb{R}, +, \cdot)$ and $(\mathbb{C}, +, \cdot)$ are *complete*.

8. Prove that every bounded sequence $(a_n)_{n\geq 1}$ has a convergent subsequence.

9. Prove that if $(a_n)_{n\geq 1}$ converges to A then so does the sequence defined by
$$b_n = \frac{a_1 + a_2 + \cdots + a_n}{n}.$$

10. (Cambridge Tripos 1903) The arithmetic mean of the products of all distinct pairs of positive integers whose sum is n is denoted by S_n. Show that
$$\lim_{n\to\infty} \frac{S_n}{n^2} = \frac{1}{6}.$$

Problem Set 2: Special Limits

1. Consider $e(n) = (1 + \frac{1}{n})^n$. Find the general term t_k $(0 \leq k \leq n)$ in the binomial expansion of $e(n)$ and hence deduce that $e(n)$ is monotonically increasing in n.

2(a) Replace each bracketed term in t_k by 1 and deduce that $e(n) < \sum_{k=0}^{n} \frac{1}{k!}$.
(b) Use part (a) to compare $e(n)$ with a suitable geometric series to show that $e(n) < 3$. Hence we may conclude that $\lim_{n\to\infty} e(n)$ exists; we denote this limit by $e \approx 2 \cdot 71828 \cdots$.

3. By assuming the convergence of the McLaurin series for e^x show that

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}.$$

4. Denote $\sum_{k=0}^{n} \frac{1}{k!}$ by s_n. Show that if $m > n$ then

$$e(m) > 1 + 1 + \frac{1}{2!}(1 - \frac{1}{m}) + \frac{1}{3!}(1 - \frac{1}{m})(1 - \frac{2}{m}) + \cdots + \frac{1}{n!}(1 - \frac{1}{m}) \cdots (1 - \frac{n-1}{m}).$$

Hence deduce that $e \geq s_n \geq e(n)$ for all $n \geq 0$ and thereby directly establish the result of Question 3.

5(a) Let $l(x) = \log_b x$. By writing $l(\frac{x}{a}) = l(x) - l(a)$ and taking the derivative of both sides, show that

$$(\log_b x)' = \frac{\lambda}{x}$$

for some constant λ.
(b) By finding $(\log_b x)'_{|x=1}$ from first principles, show that $\lambda = 1$ exactly when $b = e$.

6. Prove that if $h > 0$ then $(1 + h)^n > 1 + nh$ for all integers $n \geq 2$.

7. Let $a_n = p^{1/n}$ for some $p > 0$. We show that $a_n \to 1$ (clear for $p = 1$). For $p > 1$, write $p = 1 + h$ $(h > 0)$ and $a_n = 1 + h_n$ for some $h_n > 0$. Show using Question 6 that $a_n \to 1$. For $0 < p < 1$ show that p has the form

$$p = \frac{1}{1 + h_n},\ h_n > 0$$

and prove that $a_n \to 1$ in this case also.

Let $a_n = \sqrt[n]{n}$. We show that $a_n \to 1$ with use of the following artifice.

8. Put $b_n = \sqrt{a_n}$. We may write $b_n = 1 + h_n$ for some $h_n > 0$. Show that $h_n \leq \frac{1}{\sqrt{n}}$.

9. By writing $a_n = b_n^2$, use Question 8 to prove the claim made there.

10. Use the technique of Question 8 to prove that for any $\alpha > 1$, $\lim_{n\to\infty} \frac{n}{\alpha^n} = 0$. (Use an inequality of the form $\sqrt{\alpha^n} > 1 + nh$).

Problem Set 3: Continuous functions

1. Show that if $(a_n)_{n\geq 1} \to l$ and the function $f : \mathbb{R} \to \mathbb{R}$ is continuous at l then $(f(a_n))_{n\geq 1} \to f(l)$.

2. Show that the function $y = |x|$ is continuous.

3. Prove that if $f(x)$ is continuous at $x = a$ and $g(x)$ is continuous at $x = f(a)$ then $g(f(x))$ is continuous at $x = a$. In particular, $f(x)$ is continuous implies that $|f(x)|$ is continuous. Is the converse of this last statement true?

4. Show that if $f(x)$ and $g(x)$ are each continuous at $x = a$, then so is $h(x) = \lambda f(x) + \mu g(x)$ where $\lambda, \mu \in \mathbb{R}$.

5. Assuming that $\sin x$ and $\cos x$ are both continuous at 0, show that $f(x) = \sin x$ is continuous.

6. Let $a > 1$ and let $p(x)$ be a polynomial with real coefficients. Show using Question 10 Set 2 that

$$\lim_{n\to\infty} \frac{p(n)}{a^n} = 0.$$

7. Prove that if $f(x)$ is differentiable at $x = a$ then $f(x)$ is continuous at the same point.

8. Show that the function defined by the rule:

$$f(x, y) = \frac{x^3}{x^2 + y^2}$$

has a *removable discontinuity* at the origin by showing that the limit as $(x, y) \to (0, 0)$ exists.

9. Repeat Question 8 for the function defined by the rule:

$$f(x, y) = \frac{x^2 - y^2}{x^2 + y^2}$$

this time showing that the limit does not exist by taking limits along the line $y = mx$ as $x \to 0$.

10(a) Show that

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^4 + y^2}$$

is 0 as $(x, y) \to (0, 0)$ along any straight line $y = mx$.

(b) Show nevertheless that the discontinuity at the origin of the corresponding function is not removable by taking the limit as the origin is approached along the parabola $y = x^2$.

Problem Set 4: Series

Throughout $\Sigma = \sum_{n=1}^{\infty} a_n$ denotes the sum of a series of real numbers that may or may not converge; the partial sum $\sum_{k=1}^{n} a_k$ will be denoted by s_n.

1. Prove that if $\sum_{n=1}^{\infty} a_n$ converges then $a_n \to 0$.

2. Show that convergence of Σ is equivalent to the statement that

$$\lim_{k\to\infty} \sum_{n=k}^{\infty} a_n = 0.$$

3. Show that any ***absolutely convergent*** series (meaning that $\sum |a_n|$ converges) is itself convergent.

4. (*Alternating signs test*) Show that a series of the form $\sum_{n=0}^{\infty}(-1)^n a_n$ $(a_n \geq 0)$ converges if $(a_n)_{n\geq 0}$ converges monotonically to 0.

5. Show by the *integral test* that $\sum_{n=1}^{\infty} n^{-p}$ converges if and only if $0 < p < 1$.

6. *Ratio test* Suppose that $\lim_{n\to} |\frac{a_{n+1}}{a_n}| = r$ exists. Show Σ converges if $r < 1$ and diverges if $r > 1$.

7. Show using the Ratio test that the standard *McLaurin series* (Taylor series about 0) for e^x, $\sin x$, and $\cos x$ converge for all real x.

8. *Comparison test* Suppose that $0 \leq a_n \leq b_n$ for all $n \geq 1$. Show that $\sum a_n$ converges if $\sum b_n$ converges, and that $\sum b_n$ diverges if $\sum a_n$ diverges.

9. By using a suitable test, decide on the convergence of the following series.

$$\text{(i)} \sum_{n=1}^{\infty} \frac{2^{3n+1}}{n!} \quad \text{(ii)} \sum_{n=1}^{\infty} \frac{2^{n^2}}{(2n)!} \quad \text{(iii)} \sum_{n=1}^{\infty} \frac{\ln n}{n}.$$

10. *The root test* If $\lim_{n\to} \sqrt[n]{|a_n|} = L < 1$ then $\sum_{n=1}^{\infty} a_n$ converges absolutely (and diverges if $L > 1$). Apply the root test for these series:

$$\text{(i)} \sum_{n=1}^{\infty} \frac{(1+n^2)^{2n}}{(1-2n^2)^n} \quad \text{(ii)} \sum_{n=1}^{\infty} \frac{n^n}{5^{2+3n}} \quad \text{(iii)} \sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}.$$

Problem Set 5: Further problems on sequences and series

1. *Cauchy condensation technique* Let $(a_n)_{n\geq 1}$ be a decreasing sequence of non-negative real numbers. Show that $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{k=0}^{\infty} 2^k a_{2k}$ converges.

2. Use Question 1 to prove the result of Question 5 of Set 4.

3. For which values of p does the following series converge?

$$\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}.$$

4. Does this series diverge?

$$\sum_{n=2}^{\infty} \frac{1}{n \log n(\log(\log n))}.$$

5(a) Suppose that f is a function defined on some subset $S \subseteq \mathbb{R}$. Let $u \in S$ and suppose that for every sequence $(u_n)_{n\geq 1}$ in S such that $u_n \to u$ that $f(u_n) \to f(u)$ then f is continuous at u.
(b) Is the converse of this implication true?

6(a) Find the value of the sum $\sum_{n=1}^{\infty} \frac{n}{(n+1)!}$ by showing that

$$\sum_{n=1}^{N} \frac{n}{(n+1)!} = 1 - \frac{1}{(N+1)!}$$

(b) Alternatively, answer part (a) through finding the series for

$$\frac{1 - e^x + xe^x}{x}.$$

7(a) *Binomial series* For $f(x) = (1+x)^\alpha$ $(\alpha \in \mathbb{R})$ show that

$$(1+x)f'(x) = xf(x).$$

(b) Hence show that the series expansion of $f(x)$ is given by

$$f(x) = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n \text{ where}$$

$$\binom{\alpha}{n} = \frac{(\alpha - n + 1)(\alpha - n + 2)\cdots(\alpha - 1)\alpha}{n!}.$$

8. Show that the Binomial series converges for $|x| < 1$ and diverges for $|x| > 1$.

Our series certainly represents a function that satisfies the equation $(1 + x)f'(x) = \alpha f(x)$ with $f(0) = 1$. We now prove what our formal manipulation has suggested, which is that $f(x) = (1 + x)^\alpha$.

9. Let

$$\phi(x) = \frac{f(x)}{(1+x)^\alpha}.$$

Show that $\phi(x) \equiv 1$ and hence conclude that

$$(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n, \ \forall -1 < x < 1.$$

10. Find the form of the expansion for $\sqrt{1+x}$, writing out the expansion explicitly for term up to x^4 and use it to estimate $\sqrt{2}$.

Comment If α is a non-negative integer, of course the binomial function is a finite series that converges for all x. For all other values of α, the series is absolutely convergent for $|x| < 1$ and divergent for $|x| > 1$. For $x = 1$ the series converges absolutely if $\alpha > 0$, converges conditionally if $-1 < \alpha < 0$, and diverges if $\alpha \leq -1$. Finally, at $x = -1$ the series is absolutely convergent if $\alpha > 0$, and divergent if $\alpha < 0$.

Problem Set 6: Continuity theorems

1. Suppose that $\lim_{x\to a^+} f(x) = f(a) > 0$. Then there exists $\delta > 0$ such that $f(x) > 0$ for all $0 \le x - a < \delta$.

2. *Intermediate value theorem* Let $f(x)$ by continuous on $[a, b]$ with $f(a) < 0$ and $f(b) > 0$. There there exists α such that $a < \alpha < b$ and $f(\alpha) = 0$.

3. Show that each of the following equations have solutions:
(i) $x = \cos x$; (ii) $\sin x = x - 1$; (iii) $p(x) = 0$, where $p(x)$ is a polynomial of odd degree.

4. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a function from Euclidean n-space to Euclidean m-space. Prove that f is continuous if and only if for any open subset $U \subseteq \mathbb{R}^n$, $f^{-1}(U)$ is open in $\mathbb{R}^m$.
(b) Does this theorem hold if the word 'open' is replaced by 'closed'?

5. Prove that the mapping $f(x) = x^2$ is continuous on the real line but that there exists an open interval of $\mathbb{R}$ that is not mapped onto an open set by $f(x)$.

6. Show that if $f(x)$ is continuous at a there there exists $\delta > 0$ such that $f(x)$ is bounded on the interval $(a - \delta, a + \delta)$.

7. Prove that if f is continuous on $[a, b]$, then f is bounded on $[a, b]$.

8. *Maximum theorem* Let f be a continuous function on the closed interval $[a, b]$. Then there exists $y \in [a, b]$ such that $f(x) \le f(y)$ for all $x \in [a, b]$. Show this as follows. From Question 7 we have a least upper bound M for $f(x)$ on $[a, b]$. Take a sequence of values x_n such that $f(x_n) > M - \frac{1}{n}$. Take a convergent subsequence (why does one exist?) and show that its limit x lies in $[a, b]$ and, by contradiction, show that $f(x) = M$.

9. *Minimum theorem* By considering $-f(x)$, use the the result of Question 7 to show that there exists $z \in [a, b]$ such that $f(z) \le f(x)$ for all $x \in [a, b]$.

10. Complete the proof of the Maximum theorem as follows: suppose that $M \ne f(y)$ for all $y \in [a, b]$. Consider

$$g(x) = \frac{1}{M - f(x)}, \;\; x \in [a, b].$$

Derive a contradiction by showing that $g(x)$ is continuous on $[a, b]$ but is not bounded.

Problem Set 7: Uniform continuity

A function f on some domain of the real line is *uniformly continuous* if for all $\varepsilon > 0$ there exists $\delta > 0$ such that $|x - y| < \delta$ implies that $|f(x) - f(y)| < \varepsilon$.

1. How does this definition differ from that of *continuous* function and which condition is stronger?

2. Show that the function $f(x) = \frac{1}{x}$ on the interval $(0, 1]$ is continuous but not uniformly continuous.

3. Prove that if $f(x)$ is a continuous real-valued function on the closed interval $[a, b]$ then $f(x)$ is uniformly continuous.

Let $(f_n(x))_{n\geq 1}$ be a sequence of functions each defined on the same real domain S. We say that the $f_n(x)$ *converge pointwise* to a function $f(x)$ on S if $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in S$. We say that $f_n \to f$ *uniformly* if for any $\varepsilon > 0$ there exists N such that for all $n \geq N$ $|f_n(x) - f(x)| < \varepsilon$ for all $x \in S$. (Again, note that this is a stronger condition than pointwise convergence where the value of N not only depends on ε but also on x.)

4. Prove that the sequence $f_n(x) = \frac{1-x^n}{1-x}$ $(n = 1, 2, \cdots)$ converges uniformly to $f(x) = \frac{1}{1-x}$ on the set $S = \{x : |x| \leq a\}$ for each a such that $0 < a < 1$.

5. Let $f_n(x) = (n + 1)(n + 2)x(1 - x)^n$ $n = 1, 2, \cdots$.
(a) Show that the $f_n(x)$ converge pointwise to the zero function on the domain interval $0 \leq x \leq 1$.
(b) Is it true that

$$\int_0^1 \lim_{n\to} f_n(x)\, dx = \lim_{n\to} \int_0^1 f_n(x)\, dx?$$

(c) Show that the sequence $(f_n)_{n\geq 1}$ does not converge uniformly.

6. Suppose that $f_n \to f$ and $g_n \to g$ uniformly on some set S. Prove that:
(a) $af_n + bg_n \to af + bg$ uniformly on S;
(b) Suppose further that $|f(x)|$ and $|g(x)|$ are both bounded above on S. Show that $f_n(x)g_n(x) \to f(x)g(x)$ uniformly on S.

7(a) Suppose that the functions $f(x)$ and $|f(x)|$ are both integrable on $[a, b]$. Show that

$$\int_a^b |f(x)|\, dx \geq |\int_a^b f(x)\, dx|.$$

[Hint: you may assume that the integral of a non-negative integrable function is non-negative, a fact that follows from the definition of integral.]
(b) Left $f_n(x)$ be a sequence of integrable functions on $[a, b]$ that converge uniformly to the integrable function $f(x)$. Prove that

$$\lim_{n\to\infty} \int_a^b f_n(x)\, dx = \int_a^b \lim_{n\to\infty} f_n(x)\, dx.$$

8. By interpreting infinite sums as the limit of initial partial sums, explain what it means for a sum of the form $\sum_{k=0}^{\infty} u_k(x)$ to converge uniformly to a limiting function $u(x)$ on some interval $[a, b]$. Show that, in these circumstances

$$\sum_{k=0}^{\infty} \int_a^b u_n(x)\,dx = \int_a^b \sum_{k=0}^{\infty} u_n(x)\,dx.$$

9. Suppose that $\{f_n\}_{n\geq 1}$ is a sequence of differentiable functions on $[a, b]$ and that $f_n \to f$ pointwise. Suppose further that $\{f_n'\}$ converge uniformly on $[a, b]$ to some continuous function g. Show that f is differentiable and

$$f'(x) = \lim_{n\to\infty} f_n'(x).$$

[Hint: integrate g from a to x ($x \in [a, b]$) and apply the result of Question 7.]

10. Show that if $\sum_{k=0}^{\infty} u_k(x)$ converges pointwise to $u(x)$ and $\sum_{k=0}^{\infty} u_k'(x)$ converges to some continuous function then $u'(x) = \sum_{k=0}^{\infty} u_n'(x)$ for all $x \in [a, b]$.

Problem Set 8: Power series

1. *Weierstrass M-test* Suppose that each member of the sequence of functions $u_n(x)$ is defined for all $x \in S$. Suppose further that $\sum_{n=0}^{\infty} v_n$ is a convergent series of non-negative constants such that $|u_n(x)| \leq v_n$ for all $x \in S$. Then $\sum_{n=0}^{\infty} u_k(x)$ converges uniformly and absolutely to some function $u(x)$ on S.

2(a) Demonstrate the convergence, for any $a \in \mathbb{R}$, of the series

$$\sum_{k=0}^{\infty} \frac{a^{2k+1}}{k!}.$$

(b) Show that

$$\int_0^x e^{-t^2}\, dt = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \cdot \frac{x^{2k+1}}{2k+1}.$$

Consider a general power series

$$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n \quad a, a_n (n \geq 0) \in \mathbb{R} \tag{1}$$

Here we prove that $f(x)$ either converges only for $x = a$, or converges for all real x, or there is a postive number R, the *radius of convergence*, such that $f(x)$ converges absolutely and uniformly for all x such that $|x - a| < R$ and diverges for all x such that $|x - a| > R$. We say that $R = 0$ or $R = \infty$ respectively in the first and third cases. It is convenient to first derive these facts for $a = 0$ as follows.

3. Show that if $f(r)$ converges for some $r \geq 0$ then $f(x)$ converges absolutely on $(-r, r)$.

4. Hence show that $f(x)$ has a *radius of convergence* as described above.

5. Suppose that $a_n \neq 0$ for all but finitely many subscripts n. Then if the limit

$$R = \lim_{n\to\infty} |\frac{a_{n+1}}{a_n}|$$

exists or is $+\infty$, then R is radius of convergence of $f(x)$.

6. Show using the Weierstrass M-test that for any $r < R$, $f(r)$ converges uniformly on $[-r, r]$.

7. Show that $f(x)$ as given in (1) has an *interval of convergence* $|x - a| < R$ and $f(x)$ converges uniformly on each interval $[-r + a, r + a]$ for all $r < R$.

8. Prove that a series $\sum_{n=0}^{\infty} a_n x^n$ and it series of term-by-term derivatives $\sum_{n=1}^{\infty} n a_n x^{n-1}$ have a common radius of convergence.

9(a) Use the geometric series that converges to $\frac{1}{1+x}$ to find the series for $\log(1+x)$ centred about 0 and give its radius of convergence.

(b) From this also find the series expansion for $-\log(1-x)$, $|x| < 1$.

10. Find the sum of the series:

$$f(x) = \frac{x^2}{2} - \frac{x^3}{3 \cdot 2} + \frac{x^4}{4 \cdot 3} - \frac{x^5}{5 \cdot 4} + \cdots \quad |x| < 1$$

Problem Set 9: Mean Value Theorem and applications

1. *Rolle's theorem*: if a real-valued function f is continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b) and $f(a) = f(b)$ then there exists $c \in [a, b]$ such that $f'(c) = 0$. Prove this via the Maximum principle of Question 8 Set 6.

2. Give an example that shows that Rolle's theorem does not necessarily hold if there is one point in (a, b) where f is not differentiable.

Mean value theorem: if f is a continuous function on $[a, b]$ and differentiable on (a, b) then there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

3(a) Let $g(x) = f(x) - rx$ where r is a constant. Determine r so that $g(x)$ satisfies the conditions of Rolle's theorem.

(b) Prove the MVT by applying Rolle's theorem to $g(x)$.

4(a) Let $f(x)$ be a function that is n times differentiable at $x = a$. Show that the polyonmial:

$$P_n(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n$$

has the properties that $P_n^{(k)}(a) = f^{(k)}(0)$ for all $k = 0, 1, \cdots, n$.

Taylor's theorem: Let $f : [a, b] \to \mathbb{R}$ with $f^{(k)}(x)$ continuous on $[a, b]$ $(0 \leq k \leq n - 1)$ and suppose further that $f^{(n)}(x)$ exists on $[a, b]$. Then there exists $c \in (a, b)$ such that

$$f(b) = P_{n-1}(b) + \frac{f^{(n)}(c)}{n!}(b - a)^n.$$

(b) Show that the function

$$F(x) = f(b) - f(x) - f'(x)(b - x) - \frac{f^{(2)}(x)}{2!}(b - x)^2 - \cdots - \frac{f^{(n-1)}(x)}{(n-1)!}(b - x)^{n-1}$$ satisfies

$$F'(x) = -\frac{f^{(n)}(x)(b - x)^{n-1}}{(n-1)!}.$$

(c) Show that Rolle's theorem may be applied to

$$g(x) = F(x) - \left(\frac{b - x}{b - a}\right)^n F(a)$$

and hence deduce Taylor's theorem.

5. Use Taylor's theorem to show that

$$1 - \frac{1}{2}x^2 \leq \cos x \ \forall x \in \mathbb{R}.$$

6(a) Let $x_0 \in (a,b)$ and suppose $n \geq 2$ and $f^{(k)}(x_0) = 0$ for $k = 1, 2, \cdots n-1$, with $f^{(k)}(x)$ continuous on (a,b) for all $0 \leq k \leq n$. For any $x \in (a,b)$ show that for some c between x and x_0 we have:

$$f(x) = f(x_0) + \frac{f^{(n)}(c)}{n!}(x - x_0)^n.$$

(b) Hence deduce in these circumstances that if n is even and $f^{(n)}(x_0) > 0$, then f has a local minimum at x_0.

7. Show that for any positive integer k and positive x:

$$x - \frac{1}{2}x^2 + \cdots + \frac{x^{2k-1}}{2k-1} - \frac{x^{2k}}{2k} < \log(1+x) < x - \frac{x^2}{2} + \cdots + \frac{x^{2k+1}}{2k+1}.$$

8. *Mean value theorem for integrals* If f and g are continuous on $[a,b]$ and g does not change sign on the interval then there exists $c \in [a,b]$ such that

$$\int_a^b f(x)g(x)\,dx = f(c)\int_a^b g(x)\,dx.$$

(a) To prove this, first apply the Extreme value theorem to $f(x)$ to show either $\int_a^b g(x)\,dx = 0$ or we my write

$$m \leq \frac{\int_a^b f(x)g(x)\,dx}{\int_a^b g(x)\,dx} \leq M.$$

(b) Now deduce the theorem by applying the Intermediate value theorem to f.

9. *Symmetry of partial derivatives*: $f_{xy} = f_{yx}$. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a function with continous second partial derivatives f_{xy} and f_{yx} on some open domain R. Consider the function of x

$$A = \phi(x+h) - \phi(x), \text{ where } \phi(x) = f(x, y+k) - f(x,y).$$

(a) By applying the MVT show that A may be written as $A = h\phi'(x + \theta h)$ for some $0 < \theta < 1$.

(b) Using that $\phi'(x) = f_x(x, y+k) - f_x(x,y)$, apply the MVT to show that we may write A is the form:

$$A = hkf_{yx}(x + \theta h, y + \theta' k),$$

where $0 < \theta, \theta' < 1$.

(c) Show that we may also write A in the form:

$$A = hkf_{xy}(x + \theta_1 h, y + \theta_1' k)$$

where $0 < \theta_1, \theta_2 < 1$. Hence deduce that $f_{xy}(x, y) = f_{yx}(x, y)$.

10. Let

$$f(x, y) = xy\frac{x^2 - y^2}{x^2 + y^2},\ f(0, 0) = 0.$$

(a) Show, from first principles, that $f_x(0, y) = -y$ and $f_y(x, 0) = x$.
(b) Now show that $f_{xy}(0, 0) \neq f_{yx}(0, 0)$ although both sides are defined.

Comment This example does not contradict the theorem of Question 9 as here f_{xy} has a discontinuity at the origin. It can be shown by a little more analysis that equality of mixed partial derivatives holds as long as f_x, f_y, and *at least one of* f_{xy} and f_{yx} is continuous at the point (x, y) in question.

Problem Set 10: Lagrange and Integral remainder for Taylor series

1. *Uniqueness of series expansion* Suppose that $f(x)$ may be differentiated any number of times in some interval $I = (-r, r)$ and that $f(x) = \sum_{n=0}^{\infty} a_n x^n$. Show that $a_n = \frac{f^{(n)}(0)}{n!}$.

Recall Taylor's theorem as given in Question 4 of Set 9: Let $f : [a, b] \to \mathbb{R}$ with $f^{(k)}(x)$ continuous on $[a, x]$ $(0 \leq k \leq n)$ and suppose further that $f^{(n+1)}(x)$ exists on $[a, x]$. Then there exists $c \in (a, x)$ such that

$$f(x) = P_n(x) + R_n(x) \text{ where}$$

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}.$$

The term $R_n(x)$ is known as the *Lagrange remainder term.* The *integral form* of $R_n(x)$ is given by

$$R_n(x) = \frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)\, dt \tag{2}$$

2. Use integration by parts to verify that (2) holds for $n = 1$.

3. Prove the integral remainder formula by induction on n.

4. Apply the integral form of the remainder to show that for $f(x) = \sin x$ expanded around $a = 0$ we get:

$$|R_n(x)| \leq \frac{|x|^{n+1}}{(n+1)!}$$

and thus show the McLaurin series for $\sin x$ converges for all real x. (Note the cases where $x > 0$ and $x < 0$ are similar but should be handled separately.)

5(a) Find the Taylor polynomial $P_2(x)$ for $f(x) = \sqrt[3]{x}$ centred at $a = 8$.
(b) Use the Lagrange form of the remainder to find bounds on the accuracy of $P_2(7)$.

6(a) By substituting into the exponential series, find the McLaurin series (Taylor series centred at the origin) for $f(x) = e^{-x^2}$.
(b) Hence estimate

$$\int_0^1 e^{-x^2}\, dx$$

by integrating the first three non-zero terms of the series from (a) and obtain a bound on the error.

7. Find the first three terms of the Taylor series for $y(x)$ given that

$$x^2 + y^2 = y, \; y(0) = 1$$

Taylor series for function of two variables $f(x, y)$ with centre (a, b)

8. Suppose that

$$f(x, y) = \sum_{m,n=0}^{\infty} a_{m,n}(x - a)^m (y - b)^n$$

Extend the approach of Question 1 to partial derivatives to find an expression for the coefficient $a_{m,n}$.

9(a) Write down explicitly the form of the approximating linear polynomial of the Taylor series for $f(x, y)$ and interpret this equation geometrically.

(b) Write down the quadratic approximating (terms up to degree 2) Taylor polynomial for $f(x, y)$.

10. Find the Taylor series centred at the origin for the function of two variables

$$f(x, y) = \frac{1}{1 - x - y}.$$

Hints for Problems

Problem Set 1

Some general hints that apply to analysis problems. An inequality of the form $|a-b|<\varepsilon$ is equivalent to $-\varepsilon<a-b<\varepsilon$, a form that is sometimes easier to manipulate. When more than one variable term is involved, for example, $|(A-a_n)+(B-b_n)|$ make use of the triangle inequality $(\cdots \leq |A-a_n|+|B-b_n|\cdots)$ and deal with the limiting behaviour of each term separately.

4. Use Question 3 and that $|a_nb_n-AB| = |a_nb_n-Ab_n+Ab_n-AB| \leq \cdots$.

5. Make use of Question 3 and explain why it is enough to prove the result in the case where a_n is the constant sequence $a_n = 1$ for all n.

8. The sequence is contained in an interval $[-M, M]$. One half of this interval contains infinitely many terms. Take that subinterval and repeat the argument; then use the sequence of intervals so created to define a convergent subsequence.

10. $S_n = \frac{1}{\lfloor\frac{n}{2}\rfloor}\sum_{k=1}^{\lfloor\frac{n}{2}\rfloor} k(n-k)$; examine the cases where n is even and odd separately.

Problem Set 2

2. Replace each bracketed term of $t(k)$ by 1 to get a series bounding $e(n)$ above. Then observe that $2^{m-1} < m!$ for all $m \geq 1$ so that $\frac{1}{m!} < \frac{1}{2^{m-1}}$.

4. Since $m > n$, $e(m)$ consists of all these terms together with more.

5(a) After differentiating take $x = a$.

5(b) You may need to take a limit inside of the log function, which is justified as log is continuous. (Establishing this fact is essentially Question 1 Set 3.)

7. For $p < 1$ write $p^{\frac{1}{n}} = 1 - r_n$ and write $p^{\frac{1}{n}}$ in the form $\frac{1}{1+h_n}$.

10. Write $a_n = \frac{\sqrt{n}}{(\sqrt{\alpha})^n}$ and then write $\sqrt{\alpha} = 1 + h$ for some $h > 0$.

Problem Set 3

6. Proceed by induction on k.

7. Show that we may write $f(a+h) - f(a) = hf'(a) + h\varepsilon(h)$, where the function $\varepsilon(h) \to 0$ as $h \to 0$.

8. Recast the problem in polar co-ordinates.

Problem Set 4

1 & 2. The idea of Cauchy convergence among the sequence of partial sums is helpful here.

3. Follows using Cauchy convergence on the partial sums together with the Triangle inequality.

4. Show the sums of an even number of terms, and an odd number of terms, are respectively decreasing and increasing. Then show the corresponding limits of these series must be equal.

6. Write $2\varepsilon = 1 - r$, put $s = r + \varepsilon < 1$, show $|a_{n+1}| < s|a_n|$ for sufficiently large n and then work with the geometric series that arise.

Problem Set 5

1. Let $s_n = a_1 + a_2 + \cdots + a_n$, $t_k = a_1 + 2a_2 + \cdots + 2^k a_{2^k}$ and show that $s_n \leq t_k$ for $n \leq 2^k$ but for $n \geq 2^k$, $2s_n \geq t_k$.

3. Look to the condensed series instead.

4. Integral test works well here.

6(a) Partial fractions leads to a telescoping series.

6(b) Begin with the series for e^x, subtract 1, divide by x, differentiate term-by-term, and finaly multiply by x.

7(b) Substitute the series into the equation, differentiate term-by-term and equate coefficients.

8. Ratio test.

9. Show that $\phi'(x) = 0$.

Problem Set 6

2. Let $A = \{x : a \leq x \leq b,\ f(y) < 0 \,\forall\, a \leq y \leq x\}$. Show that the least upper bound α of A exists and show by contradiction that $f(\alpha) = 0$.

3. Apply the IVF.

7. Let $A = \{x : a \leq x \leq b$ and f is bounded on $[a, x]\}$. Use Question 6 to show that the least upper bound of A is b. Deduce that f is bounded on all intervals of the form $[a, b - \delta]$ $(\delta > 0)$. Use symmetry to gain the same conclusion for $[a + \delta, b]$, from where the result follows easily.

8. Let M be the least upper bound for f on $[a, b]$ and take a sequence of points in $[a, b]$ whose images under f become arbitrarily close to M. Take the limit x of a convergent subsequence and consider the value of $f(x)$.

9. Apply the result of Question 8 to $-f(x)$. Then show that $-f$ has a maximum value of m if and only if f has a minimum of $-m$.

10. This trick is often useful: by assuming that M is never attained by f it follows that g is continuous on $[a, b]$ and therefore bounded, but this quickly leads to contradiction.

Problem Set 7

3. Suppose $f(x)$ were not uniformly continous. Let $(\delta_n)_{n \geq 1}$ decrease monotonically to 0. Then for each δ_n there exists $x_n, y_n \in [a, b]$ such that $|x_n - y_n| < \delta_n$ but $|f(x_n) - f(y_n)| > \varepsilon$. Take a convergent subsequence of the x_n that approaches some limit $x \in [a, b]$. The use the continuity of f to gain a contradiction.

6(b) Consider $|f_n(x)g_n(x) - f(x)g(x)| = |f_n(x)g_n(x) - f_n(x)g(x) + f_n(x)g(x) - f(x)g(x)|$.

8. The function defined by the infinite sum is the limit of the functions defined by the partial sums.

10(b) Begin by substituting in the standard exponential series.

Problem Set 8

1. Use the dominating series to show that the tail of the sum of the sequence of functions may be made arbitrarily small, independently of x.

2(b) Justify taking the infinite sum inside the integral by uniform continuity, verified through the M-test making use of the series in (a).

Problem Set 9

1. Either f is constant or it has a local maximum or minimum, c. What is $f'(c)$?

2. Try the absolute value function.

5. Treat the interval $[-\pi, \pi]$ and its complement separately.

Problem Set 10

2. & 3. Integration by parts.

5. This is an alternating series with terms of monotonically decreasing magnitude so that the absolute value of each term is itself an error bound.

7. Use implicit differentiation.

8. Look at $\frac{\partial f^{m+n}}{\partial f^m \partial^n}|_{(x,y)=(a,b)}$.

9. Subscript notation for partials is easier to use here.

Answers to the Problems

Problem Set 4

9(i) convergent; (ii) divergent; (iii) divergent. 10(i) divergent; (ii) divergent; (iii) convergent.

Problem Set 5

3. $p > 1$. 4. Yes. 6. 1. 10. $1 + \frac{1}{2}x - \frac{1}{2\cdot 4}x^2 + \frac{1\cdot 3}{2\cdot 4\cdot 6}x^3 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}x^4 + \cdots$; $\sqrt{2} \approx 1\frac{51}{128} = 1 \cdot 40\,(2\,\text{d.p})$.

Problem Set 7

5(b) No, LHS $= 0$, RHS $= 1$ (by integration by parts)

Problem Set 8

9(a) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n}$, $\forall\,|x| < 1$; (b) $x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots$. 10. $f(x) = (1+x)\log(1+x) - x$.

Problem Set 10

8. $1-x^2$. 9. $\therefore a_{m,n} = \frac{1}{m!n!}\frac{\partial f^{m+n}}{\partial^m x \partial^n y}|_{(x,y)=(a,b)}$. 10. $\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(m+n)!}{m!n!}x^m y^n$.