Mathematics 105 Probability & Combinatorics Solutions

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Solutions and Comments for the Problems

Problem Set 1

1. The constant term is $\binom{10}{5}(-\frac{1}{2})^5 = \frac{10\cdot 9\cdot 8\cdot 7\cdot 6}{5\cdot 4\cdot 3\cdot 2\cdot 2^5}(-1)^5 = -\frac{63}{8}$.

2. $P(\text{player misses}) = 1 - P(\text{he does not miss}) = 1 - 0\cdot 8 = 0\cdot 2$. Hence $P(\text{two successive misses}) = (0\cdot 2)^2 = 0\cdot 04$.

Comment Of course this calculation is implicitly assuming that the outcome of the second shot is independent of the first. This is justified as the question says plainly that the success probability is 0.8, and so we are within our rights to assume that this is true in all circumstances. If it wasn't, we would need to know more before we could answer the question.

3. The Nuffing score takes on $m+1$ values throughout the match $(0, 1, \cdots, m)$ and similarly the Plough score takes on $n+1$ values, any of which may have applied at half time, yielding a total of $(m+1)(n+1)$ possibilities.

4. The product $(m+1)(n+1)$ 'double counts' all the pairs $\{k, l\}$ with $k \neq l$. The number of such pairs is $\binom{n+1}{2}$ (as $m \geq n$) , giving the answer:

$$(m+1)(n+1) - \frac{1}{2}n(n+1) = \frac{1}{2}(n+1)(2m-n+2).$$

5. There are $3! = 6$ possibilities; there is 1 way of getting all 3 correct; there is no way of getting 2 (and not all 3) correct; there are 3 ways of getting just one right. Hence there are $6-(1+3) = 6-4 = 2$ ways of being dead wrong. Hence $Pr(\text{none correct}) = \frac{2}{6} = \frac{1}{3}$.

Comment This solves this problem but we are immediately led to think how we might try and solve it if instead of just 3 leads and 3 sockets there were some arbitrary number n. This problem is solved through an application of what is called the *Inclusion-Exclusion Principle.* The denominator of the probability quotient is easily seen to be $n!$ as that is how many permutations of leads and sockets is possible. The Principle then allows us to find the numerator through a finite series with alternating signs that represents a series of corrections and reverse correction terms. The outcome in this case is equal to the first $n+1$ terms of the series for $e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^n\frac{1}{n!} + \cdots$. Since this series converges very quickly, for even modest values of n the answer is as near as makes no difference to e^{-1}, which corresponds to approximately $36\cdot 7\%$. Another instance of this problem is where you take two well-shuffled packs of cards and turn over one from each pack and compare. The probability that there is no match is therefore almost exactly 0.367.

6. It is perhaps easier to count the 2×2, 3×3, etc. squares by counting the number of their centres. This yields, not surprisingly perhaps, a sum of squares as there are 8^2 unit squares, 7^2 squares of the 2×2 variety and so on to a single 8×8 square.

$$8^2 + 7^2 + \cdots + 1^2 = 204.$$

7. A rectangle on the chessboard is determined by a choice of 2 horizontal lines (from $\underline{9}$) and 2 vertical lines.

$$\therefore \#\text{rectangles}= \binom{9}{2}\binom{9}{2}.$$

Hence, by using the answer to Question 6 that counted the number of squares on a chessboard to be 204, we find that

$$\#\text{oblongs}= \#\text{rectangles-}\#\text{squares}=\left(\frac{9!}{7!2!}\right)^2 - \#\text{squares}= 36^2 - 204 = 1092.$$

8.

$$\int_0^1 f(x)\,dx = [\frac{kx^3}{3} - \frac{kx^4}{4}]_0^1 = \frac{k}{12} = 1.$$

Hence $k = 12$.

Comment This is the unique value of k that makes $f(x)$ into a probability density function.

9. Each player, except the eventual champion, loses precisely one match and each match has precisely one loser. Hence there is a one-to-one correspondence between the set of matches and the set of losing players. Therefore there are $n-1$ matches in the entire tournament.

10. There are $\binom{n}{2}$ possible pairings, of which $n-1$ will get to play (by Question 9). Hence the required probability is:

$$\frac{n-1}{\binom{n}{2}} = \frac{n-1}{\frac{1}{2}n(n-1)} = \frac{2}{n}.$$

Problem Set 2

1. This is the sequence of the total number of points possible in a game of tennis: after 6 points we have 'deuce' and the number of remaining points is always a multiple of 2.

2. The probability of a 'draw' is $\frac{1}{6}$. Hence, by symmetry, the probability that the green die wins is $\frac{1}{2}(1-\frac{1}{6}) = \frac{1}{2}\cdot\frac{5}{6} = \frac{5}{12}$.

3. Pr(failure by six-dice man) = Pr(no aces)= $\left(\frac{5}{6}\right) \approx 0\cdot 335$ so his success probability is $\approx 1 - 0\cdot 335 = 0\cdot 665$. On the other hand, Pr(failure by 12-dice man) is the sum of the probability of no aces, which is $\left(\frac{5}{6}\right)$ plus Pr(exactly one ace), which is $\binom{12}{1}\times\frac{1}{6}\times\left(\frac{5}{6}\right)$ (the number of aces is a binomial random variable with success probability of $\frac{1}{6}$). This gives a success probability of:

$$1 - \left(\frac{5}{6}\right) - 12 \times \frac{1}{6} \times \left(\frac{5}{6}\right) \approx 0.619.$$

Therefore it is the player with the six dice who has a slight advantage.

Comment The probability of the six-dice man succeeding while the 12-dice man fails is $0 \cdot 665 \times (1 - 0 \cdot 619) = 0 \cdot 665 \times 0 \cdot 381 = 0.253$; the probability of the 12-dice man getting an ace while the six-dice man does not is similarly $0 \cdot 619 \times 0 \cdot 335 = 0 \cdot 207$. If they play n times therefore the expected number of more times that six-dice man wins is $(0 \cdot 253 - 0 \cdot 207)n = 0 \cdot 046n$. In other words he has a gambling advantage of about 4.6% over his opponent.

4. Pr(no diamond) $= \frac{39}{52} \cdot \frac{38}{51}$. Hence Pr(at least one diamond)$= 1 - \frac{3}{4} \cdot \frac{38}{51} = 1 - \frac{19}{34} = \frac{15}{34}$.

5. Let r denote the radius of the coin. The centre of the coin C settles in some square and may equally lie any place within that square. The coin covers a corner if and only if C lies within a circle of radius r centred at some corner of the square. This area consists of four quarter circles, one at each corner. Hence the required probability is equal to:

$$\frac{\text{Area of a circle of radius } r}{\text{Area of a square of side } 2r} = \frac{\pi r^2}{(2r)^2} = \frac{\pi}{4} \approx 0.79 \text{ (to 2 decimal places).}$$

6. There are two approaches, the first using geometric series: Pr(Player 1 'wins' on $(n+1)$st turn) is $\left(\frac{5}{6} \cdot \frac{5}{6}\right)^n \cdot \frac{1}{6}$. This gives:

$$\text{Pr(Player 1 `wins')} = \tfrac{1}{6} \textstyle\sum_{n=0}^{\infty} \left(\frac{25}{36}\right)^n = \frac{1}{6} \cdot \frac{1}{1-\frac{25}{36}} = \frac{1}{6} \cdot \frac{36}{11} = \frac{6}{11}.$$

The second approach exploits the near symmetry inherent in the problem to glean the solution with little calculation. Let the players be A and B with a and b denoting their respective probabilities of winning. Now $a + b = 1$. If the first shot in the contest is fatal (probability 1/6), B can no longer win. If A survives the first shot however (probability 5/6), the tables are turned and, in effect, A and B have now swapped places with B having the 'advantage'. In other words, in the event that the first shot is a blank, the probability that B will yet go on to win is a. This gives an equation relating a and b: $b = \frac{5}{6}a$. Coupling this with the fact that $b = 1 - a$ we obtain:

$$1 - a = \frac{5}{6}a \Rightarrow 1 = \frac{11}{6}a \Rightarrow a = \frac{6}{11}.$$

7. The coefficient of $(2x)^5(-y^2)^4$ is $\binom{9}{5}$, yielding as the coefficient of x^5y^8:

$$\frac{9!}{5!4!} \cdot 2^5 \cdot (-1)^4 = \frac{9 \times 8 \times 7 \times 6}{4 \times 3 \times 2} \cdot 2^5 = 9 \times 7 \times 2^6 = 63 \times 64 = 4032.$$

8.
$$k\binom{n}{k} = \frac{k \cdot n!}{(n-k)!k!} = n\frac{(n-1)!}{(n-k)!(k-1)!} = n\binom{n-1}{k-1}$$

It follows that

$$\sum_{k=0}^{n} k\binom{n}{k} = n\sum_{k=0}^{n}\binom{n-1}{k-1} = n\sum_{k=0}^{n-1}\binom{n-1}{k} = n2^{n-1}.$$

Comment These manipulations are valid under the convention that $\binom{n}{k} = 0$ for all values of $k < 0$. In particular the re-indexing of the sum can take a lower limit of $k = 0$ rather than $k = -1$, which results from a formal change of variable from k to $k - 1$.

9. The probability of a score of $5, 4, 3, 2, 1$ and 0 by the player is respectively $\frac{2}{36}, \frac{4}{36}, \frac{6}{36}, \frac{8}{36}, \frac{10}{36}, \frac{6}{36}$. The expected win of the Bank in pounds is given by:

$$2 - \frac{1}{36}(2 \times 5 + 4 \times 4 + 6 \times 3 + 8 \times 2 + 10 \times 1 + 6 \times 0) =$$

$$= 2 - \frac{1}{36}(10 + 16 + 18 + 16 + 10 + 0) = 2 - \frac{70}{36} = \frac{2}{36} = \frac{1}{18}.$$

And so the Bank does indeed has a net expected gain of £0.055 per roll.

10. Under this playing regime, the $(6, 6)$ roll is effectively discounted. The calculation is then identical to before except the divisor of 36 is replaced by 35 (and the multiplier 6 of 0 is replaced by 5, which has no effect). Hence the expected gain of the Bank is now $2 - \frac{70}{35} = 0$ and the game is fair.

Problem Set 3

1. Pr(doubles or 8)=Pr(doubles)+Pr(8) - Pr(doubles & 8))

$$= \frac{6}{36} + \frac{5}{36} - \frac{1}{36} = \frac{5}{18}.$$

2.

$$\binom{3}{2}\frac{5}{6} \cdot \frac{1}{6} = \frac{5}{12}.$$

3.

$$2 \cdot (\binom{5}{4} + \binom{5}{5})(\frac{1}{2})^5 = \frac{12}{32} = \frac{3}{8}.$$

4. One example 11100010.

Comment Such strings, known as *de Bruijn strings*, exist for any power of 2 and have a variety of applications and interesting properties.

5. The common sum must be $\frac{1}{3} \cdot \frac{1}{2} \cdot 12 \cdot 13 = 26$. Both lines start from between 10 and 11 with the top line passing between 2 and 3 while the bottom line passes between 8 and 9 giving the partition: $\{11, 12, 1, 2\}$, $\{5, 6, 7, 8\}$ and $\{3, 4, 9, 10\}$.

6. There are 9 letters but 4 are E and 2 are V so the total number of distinguishable permutations is:

$$\frac{9!}{4!2!} = \frac{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5}{2} = 7,560.$$

7. We can choose the 'first' team in $\binom{12}{6}$ ways, but since the order of the 2 teams is immaterial, we have to divide this number by 2!, which gives:

$$\frac{12!}{2!6!6!} = \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7}{2 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2} = 11 \cdot 3 \cdot 2 \cdot 7 = 11 \times 42 = 462.$$

8. Ignoring the Higgins's for the moment, the number of admissable committees is

$$\binom{6}{2}\binom{4}{2} + \binom{6}{3}\binom{4}{1} + \binom{6}{4}\binom{4}{0} = (15)(6) + (20)(4) + (15)(1) = 185.$$

From this we subtract the number of committees that have both Higgins's on them, which numbers

$$\binom{5}{1}\binom{3}{1} + \binom{5}{2}\binom{3}{0} = (5)(3) + (10)(1) = 25,$$

$$\therefore \text{the number of allowable committees is } 185 - 25 = 160.$$

9.

$$\binom{8}{2} = \frac{8 \cdot 7}{2} = 28.$$

10. There are four equally probably possibilities for the first two tosses, which are HH,HT, TH, and TT. In the first three cases, HTT must appear before TTH is possible, while in the final case TTH must appear first. Hence the probability of the respective probabillities of the player and the bank winning are $\frac{1}{4}$and $\frac{3}{4}$ and so the expected loss of the player is

$$\frac{3}{4}(10) - \frac{1}{4}(20) = \frac{10}{4} = \frac{5}{2};$$

therefore the average 'winnings' of the player per game is $-£2 \cdot 50$.

Problem Set 4

1. $(5+5)! = 10! = 3,628,800$.

2. Each linear arrangement gives 10 equivalent cyclic arrangements, so the answer is $\frac{10!}{10} = 9! = 36,280$.

3. Arrange the girls and boys alternating in a row: this can be done in $2 \times (5!)^2 = 2 \times 120^2 = 28,800$ ways. (The factor of 2 counts whether we begin with a boy or a girl.) Divide by 10 to get the number of cyclic arrangements: $\frac{28,800}{10} = 2,880$.

4. The number of linear arrangements is

$$\binom{10}{5} = \frac{10!}{5!5!} = \frac{10 \times 9 \times 8 \times 7 \times 6}{5 \times 4 \times 3 \times 2} = 252.$$

5. The probability of $n-1$ tails followed by a head is clearly $\frac{1}{2^n}$.
6. The mean of the random variable X of Question 5 is:

$$\sum_{n=1}^{\infty} \frac{n}{2^n} = 2.$$

Comment: For $|x| < 1$ $\sum_{n=0}^{\infty} x^n = (1-x)^{-1} \Rightarrow \sum_{n=0}^{\infty} nx^{n-1} = (1-x)^{-2} \Rightarrow \sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}$. Putting $x = \frac{1}{2}$ gives the previous sum. Since we shall soon be needing it, let us also differentiate again:

$$\sum_{n=1}^{\infty} n(n-1)x^{n-2} = 2(1-x)^{-3}$$

$$\Rightarrow \sum_{n=0}^{\infty} n(n-1)x^n = \frac{2x^2}{(1-x)^3} \quad |x| < 1. \tag{1}$$

7. The event E_n we seek is that exactly one of the first $n-1$ tosses is a head, as is the nth toss. Hence

$$P(E_n) = p_n = \binom{n-1}{1} \frac{1}{2^n} = \frac{n-1}{2^n}.$$

8. We want the expectation $E(X)$ where $P(X = p_n)$ so we require:

$$\sum_{n=0}^{\infty} \frac{n(n-1)}{2^n} = \frac{2 \cdot (\frac{1}{2})^2}{\frac{1}{2^3}} = 4.$$

where we have invoked (1) with $x = \frac{1}{2}$.

9. There is a one-to-one correspondence between solutions to the equation and the number of arrangements of 12 crosses and 3 slashes (the four lists of crosses so created corresponds to the values gives to each of the x_i in turn). Hence the answer is

$$\binom{12+3}{3} = \binom{15}{3} = \frac{15 \cdot 14 \cdot 13}{6} = 5 \cdot 7 \cdot 13 = 455.$$

10. Put $x_i = y_i + 1$ so that $x_i \geq 0 \Leftrightarrow y_i \geq 1$. Substituting accordingly we get:

$$(y_1 + 1) + (y_2 + 1) + (y_3 + 1) + (y_4 + 1) = 12$$

$$\Rightarrow y_1 + y_2 + y_3 + y_4 = 8;$$

by the argument of the previous question the number of distinct solutions is

$$\binom{8+3}{3} = \binom{11}{3} = \frac{11 \cdot 10 \cdot 9}{6} = 11 \cdot 5 \cdot 3 = 165.$$

Comment: in conclusion concerning Questions 9 and 10, there are 455 solutions to the equation in non-negative integers, 165 of which contain no zeros.

Problem Set 5

1. By definition of conditional probability we have the two expressions $P(A|B) = \frac{P(A\cap B)}{P(B)}$ and by the same token $P(B|A) = \frac{P(A\cap B)}{P(A)}$. Cross multiplying gives two expression for $P(A \cap B)$:

$$P(A \cap B) = P(A|B)P(B) = P(B|A)P(A)$$

$$\Rightarrow P(A|B) = \frac{P(B|A)P(A)}{P(B)}.$$

2. Of the $6 \times 6 = 36$ equally likely outcomes of a two-dice roll, $6+6-1 = 11$ of them satisfy the condition 'at least one die shows 6'. Of these, those that sum to 9 or more can be listed as $(3, 6), (4, 6), (5, 6), (6, 6), (6, 5), (6, 4), (6, 3)$, which number 7 in all, giving the required probabilty as $\frac{7}{11}$. Alternatively, let A be the event that 'the sum of the faces exceeds 8' and B be the event 'at least one 6 is rolled'. Then $P(A\cap B) = \frac{7}{36}$ while $P(B) = \frac{11}{36}$, yielding $P(A|B) = \frac{7}{36} \cdot \frac{36}{11} = \frac{7}{11}$.

3. Let p be the unknown proportion. The given information translates to the equation:

$$(0 \cdot 5)(0 \cdot 6) + p(1 - 0 \cdot 6) = 0 \cdot 4$$

$$\Rightarrow p = \frac{0 \cdot 4 - 0 \cdot 3}{0 \cdot 4} = \frac{0 \cdot 1}{0 \cdot 4} = 0 \cdot 25.$$

4. Let A be the event 'Voter supports our party' and let B stand for 'Voter owns a Bike'. We then have

$$P(B) = P(B\cap A)+P(B\cap A') = (0\cdot 5)(0\cdot 6)+(0\cdot 25)(1-0\cdot 6) = 0\cdot 3+0\cdot 1 = 0\cdot 4.$$

Hence we obtain the required probability as

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)} = \frac{(0 \cdot 5)(0 \cdot 6)}{0 \cdot 4} = \frac{0 \cdot 3}{0 \cdot 4} = 0 \cdot 75,$$

and so her chances of finding a supporter increases from 60% to 75% if she concentrates our houses where she spots that the owner has a cycle.

5. Let W and M be the events, W = 'voter is a woman' and M = 'voter is a man', with $P(W) = P(M) = \frac{1}{2}$. Let E be the event 'person selected voted ECP'. Then we are given $P(E|M) = 0 \cdot 35$ and $P(E|W) = 0 \cdot 45$. Also since every voter is either a man or a woman but not both we have:

$$P(E) = P(E\cap W)P(W)+P(E\cap M)P(M) = (0\cdot 45)(0\cdot 5)+(0\cdot 35)(0\cdot 5) = 0\cdot 4.$$

Finally we want $P(W|E)$ and so by Bayes's Rule we obtain:

$$P(W|E) = \frac{P(E|W)P(W)}{P(E)} = \frac{(0 \cdot 45)(0 \cdot 5)}{0 \cdot 4} = 0 \cdot 5625.$$

Hence the percentage probability that the voter is a woman given that they voted ECP is 56.25%.

6. Let R and B denote the respective events, $R =$'It is raining', $B =$ 'barometer indicates rain'. We require $P(R|B)$. We are given $P(B|R) = 0 \cdot 7$ and $P(B| \sim R) = 0 \cdot 1$ and $P(R) = \frac{1}{3}$. Now

$$P(B) = P(B \cap R) + P(B \cap (\sim R)) = P(B|R)P(R) + P(B|(\sim R))P((\sim R)) =$$

$$\left(\frac{7}{10} \times \frac{1}{3}\right) + \left(\frac{1}{10} \times \frac{2}{3}\right) = \frac{7}{30} + \frac{2}{30} = \frac{9}{30} = \frac{3}{10}.$$

7. By Bayes and using Question 6:

$$P(R|B) = \frac{P(B|R)P(R)}{P(B)} = \frac{\frac{7}{10} \times \frac{1}{3}}{\frac{3}{10}} = \frac{7}{30} \times \frac{10}{3} = \frac{7}{9}.$$

8. Let $X_j \equiv$ 'selection from urn j' $(j = 1, 2, 3)$ and let $A \equiv$ 'white ball and red ball chosen from urn'. We require $P(X_2|A)$. We have $P(X_j) = \frac{1}{3}$. Also

$$P(A|X_1) = \frac{1}{6} \cdot \frac{3}{5} = \frac{1}{10}, \; P(A|X_2) = \frac{3}{5} \cdot \frac{1}{4} = \frac{3}{20}, \; P(A|X_3) = \frac{3}{9} \cdot \frac{3}{8} = \frac{1}{8}.$$

Hence

$$P(A) = \frac{1}{10} \cdot \frac{1}{3} + \frac{3}{20} \cdot \frac{1}{3} + \frac{1}{8} \cdot \frac{1}{3} = \frac{1}{3}\left(\frac{1}{10} + \frac{3}{20} + \frac{1}{8}\right) = \frac{1}{8}.$$

Therefore by Bayes's Rule:

$$P(X_2|A) = \frac{P(A|X_2)P(X_2)}{P(A)} = \frac{\frac{3}{20} \cdot \frac{1}{3}}{\frac{1}{8}} = \frac{8}{20} = \frac{2}{5}.$$

9. Let $+$ denote the event that a person tests positive and let D denote the event that the person has the disease. We are given that $P(+|D) = 1$, $P(+|\neg D) = 0 \cdot 05$. Now

$$P(+) = P(+ \,\&\, D) + P(+ \,\&\, (\neg D)) = P(D)P(+|D) + P(\neg D)P(+|\neg D) =$$

$$= 0 \cdot 001 \times 1 + 0 \cdot 999 \times 0 \cdot 05 = 0 \cdot 05095.$$

Hence

$$P(D|+) = \frac{P(+|D)P(D)}{P(+)} = \frac{1 \times 0 \cdot 001}{0 \cdot 05095} = \frac{1}{50 \cdot 95} = 0 \cdot 0196 \approx 2\%.$$

Comment Questions like this are sometimes given to people, including medical professionals, and invariably those questioned tick the box with a much higher probability. A common response is 95% as 'the test is 95% accurate'. Probability is a peculiar in that, unlike other branches of mathematics, raw intuition often leads to estimates that are wildly wrong but at the same time those making the bad guesses are convinced the problem is simple and that their answer is obviously right.

10. Let F and S be the respective events that the first and the second marble chosen were red. Then

$$P(F|S) = \frac{P(F\&S)}{P(S)} = \frac{P(F) \cdot P(S|F)}{P(S)} = \frac{\frac{2}{5} \cdot \frac{1}{4}}{\frac{2}{5}} = \frac{1}{4}.$$

Comment: We could argue that $P(F|S) = P(S|F)$ (as $P(F) = P(S)$) and the latter is clearly $\frac{1}{4}$.

Problem Set 6

Let a and b be the respective probabilities that P is absorbed at $x = -1$ and $x = 2$. Since absorption is inevitable (as the probability of endless oscillation between $x = 0$ and $x = 1$ is 0) we have $a + b = 1$. Either P initially moves left from the origin, we write this as $P \to -1$, (probability $\frac{1}{2}$) and so P is absorbed at $x = -1$, or $P \to 1$ (probability $\frac{1}{2}$). By symmetry, the probability of P being absorbed at $x = 2$ given $P \to 1$ is equal to a. Hence

$$b = Pr(P \to 1)Pr(P \text{ is absorbed at } x = 2|P \to 1) = \frac{1}{2}a;$$

substituting $a = 2b$ in our original probability equation now gives

$$2b + b = 3b = 1 \text{ and so } b = \frac{1}{3}.$$

In conclusion, the chances that the particle will eventually be absorbed at the right hand barrier is $1/3$.

2. The given information implies that

$$\frac{6}{n} \cdot \frac{5}{n-1} = \frac{1}{3} \Rightarrow n(n-1) = 3 \cdot 30$$

$$\Rightarrow n^2 - n - 90 = 0 \Rightarrow (n-10)(n+9) = 0$$

and taking the positive solution gives $n = 10$, so that is the number of sweets in Charlotte's bag.

3. Let us look at the general situation where we begin with $2n$ teams and their names are drawn from a hat, one after another, to produce n pairs. The number of different permutations that may be formed of the $2n$ names as we draw them out is $(2n)!$. Each particular drawing of the teams into pairs arises from $n!2^n$ of these permutations: the $n!$ term counts the number of arrangements of the n pairs and for each of the n pairs there is a factor of 2 corresponding to the order of draw within the pair. Hence the number of ways of splitting a set of size $2n$ into pairs is

$$p_n = \frac{(2n)!}{2^n n!}.$$

By multiplying each term in the $n!$ product be one of the instances of 2 in the 2^n term we see that the denominator can be written as

$$(2n)(2n-2)(2n-4)\cdots 2,$$

the product of the first n even numbers. Cancelling these terms into the numerator $(2n)!$ tells us that p_n, the number of ways of splitting $2n$ objects into pairs is the product of the first n odd numbers:

$$p_n = (2n-1)(2n-3)\cdots 3.$$

In this problem, $n = 8$. The total number of ways the 16 teams may be drawn into pairs is p_8. We are interested in the number of these pairings where 4 named teams are drawn together. There are is $p_2 = 3$ ways these 4 can be drawn against one another - once you pick one of the three possible opponents for any specific team, the draw is fixed for all four of them, while the other 12 teams can be drawn against one another in p_6 ways. The total number of ways therefore of splitting the 8 pairs up in this fashion is p_2p_6. Hence the required probability, p, is given by the ratio

$$p = \frac{p_2p_6}{p_8} = \frac{4!}{2^2 2!}\cdot\frac{12!}{2^6 6!}\cdot\frac{2^8 8!}{16!} = \frac{4\times 3}{1}\cdot\frac{8\times 7}{16\times 15\times 14\times 13} = \frac{1}{65}.$$

4. The total number of possibilities in the $2n$ tosses is 2^{2n}. On the other hand the number of ways in which each of the two people toss and equal number, k, of heads is $\sum_{k=0}^{n}\binom{n}{k}^2 = \binom{2n}{n}$. Hence, the required probability is:

$$\frac{\binom{2n}{n}}{2^{2n}} = \frac{(2n)!}{(2^n n!)^2} = \frac{(2n-1)(2n-3)\cdots 1}{2n(2n-2)(2n-4)\cdots 2}.$$

Comment Using the so-called Wallis Formula for π it can be shown that this probability ratio approaches $\frac{1}{\sqrt{\pi n}}$ for large n.

5. This pretty problem has arisen in different guises before but this version featured on the website of the American Mathematical Society in 2015 and triggered a viral surge on the internet. The question does not demand that the two groups are equal but we can supply such a solution.

First, split the coins into two equal groups, the left and the right say, of n coins each. The left hand group will have some unknown number, k of heads (and $n-k$ tails), for some number k in the range $0 \le k \le n$. Since we know there are n heads overall, it follows that the right hand group of coins must have the remaining $n-k$ heads (and k tails).

If we now flip over *all* the coins in the right hand group, that group will have k heads (and $n-k$ tails). Therefore both groups now have an equal number, k of heads and also an equal number, $n-k$ of tails.

Of course, we do not know the value of k, although it is not a difficult exercise to find the probability for each value of k in the experiment.

6. Let the probablilty of no tremor in the next half hour be p. Then the probability of no tremor in the next hour is $p^2 = 1 - 0\cdot 4 = 0\cdot 36$. Hence

$p = 0 \cdot 6$ and therefore the probability of at least one tremor in the next half hour is $1 - p = 1 - 0 \cdot 6 = 0 \cdot 4$.

Comment I have read that *Google* have used a version of this problem as an interview question for prospective employees - it is short but quite tricky!

7. The probability that any particular set is chosen by A is $\frac{1}{2^n}$. The probablity that A, (we use the same symbol for both set and player), has exactly k members is therefore $\frac{\binom{n}{k}}{2^n}$. The probability that $B \subseteq A$ is then $\frac{2^k}{2^n} = 2^{k-n}$. Hence, the overall probability that $B \subseteq A$ is

$$\sum_{k=0}^{n} \binom{n}{k} 2^{-n} \cdot 2^{k-n} = 4^{-n} \sum_{k=0}^{n} \binom{n}{k} 2^k.$$

Now $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$ so upon putting $x = 2$ we get the previous probablility is equal to:

$$4^{-n}(1+2)^n = \left(\frac{3}{4}\right)^n.$$

8 & 9. We introduce a fresh argument that allows us to answer Questions 7,8 and 9 together. Let the underlying set $S = \{1, 2, \cdots, n\}$. Let both A and B choose their random sets by tossing a coin n times with i included in their set if the ith toss is a head, denoted by 1 and i is excluded in the event of a tail, denoted by 0. This produces n independent experiements, the outcome of which can be coded as a binary pair, ab of four possible types, with $a = 0$ or 1 according as A tosses head or tail, and similarly for the second digit as regards B. Then we will get $B \subseteq A$ if and only if the forbidden outcome of 01 is avoided on all occasions. The probability of avoidance of the forbbidden pair is evidently $\frac{3}{4}$ so the probability that $B \subseteq A$ is $\left(\frac{3}{4}\right)^n$.

The answers to Questions 8 and 9 are the same as Question 7: in Question 8 the forbidden pair is 11 and for Question 9 it is 00.

Alternatively we note that the logical equivalence of the three events $A \subseteq (\sim B)$, $A \cap B = \emptyset$ and $(\sim A) \cup (\sim B) = S$. Since $P(A \subseteq B) = P(A \subseteq (\sim B)))$ we see that Questions 7 and 8 have the same answer. Similarly since $P((\sim A) \cup (\sim B)) = S) = P((A \cup B) = S)$ we obtain the equality of the answers to Question 8 and 9.

10. We extend the previous approach to that of k subsets $A_1, \cdots, A_k$, which are determined by coin toss so that the outcome at each stage of the construction is a binary k-tuple. The admissable k-tuples at each stage are then exactly those with no more that one instance of 1, which number $1 + k$. The 1 corresponds to the k-tuple of zeros while there are k choices of k-tuples that feature exactly one instance of 1. Hence the probability that the k-tuple chosen at any particular stage is admissable is $\frac{1+k}{2^k}$ and therefore the probability that the k-sets chosen at random are pairwise disjoint is $\left(\frac{1+k}{2^k}\right)^n$

Problem Set 7

1. The underlying random variable X here is distributed $\mathrm{Bin}(8, 0\cdot 6)$, a binomial distribution with $n = 8$ and success probability $p = 0\cdot 6$ and complementary probability $q = 1 - 0\cdot 6 = 0\cdot 4$. Hence

$$P(X > 5) = P(X = 6) + P(X = 7) + P(X = 8)$$

$$= \binom{8}{6}(0\cdot 6)^6(0\cdot 4)^2 + \binom{8}{7}(0\cdot 6)^7(0\cdot 4)^1 + \binom{8}{8}(0\cdot 6)^8(0\cdot 4)^0$$

$$= 28(0\cdot 046656)(0\cdot 16) + 8(0\cdot 0279936)(0\cdot 4) + 1(0\cdot 0167961)(1)$$

$$= 0\cdot 315 \text{ to 3d.p.}$$

2. Here $X \sim \mathrm{Bin}(n, \frac{1}{4})$ so that $q = 1 - p = \frac{3}{4}$, where X is the random variable denoting the number of red flowers chosen. Here n is unknown. However we know that $P(X = 0) < 1 - 0\cdot 95 = 0\cdot 05$. Hence we require the least n such that

$$\binom{n}{0}(0\cdot 25)^0(0\cdot 75)^n < 0\cdot 05$$

$$\Rightarrow n\log_{10}(0\cdot 75) < \log_{10}(0\cdot 05)$$

$$\Rightarrow n(-0\cdot 125) < -1\cdot 301$$

$$\Rightarrow n > \frac{1\cdot 301}{0\cdot 125} \text{ (direction of inequality flips!)}$$

so that $n > 10\cdot 4$ and since n is the least integer satisfying this inequality we conclude that $n = 11$.

3 (a) Here we are dealing with $X \sim \mathrm{Bin}(3, 0\cdot 7)$ as there are $n = 3$ identical Bernoulli trials with success probability $p = \frac{7}{10}$. We want

$$P(X = 3) = p^3 = (0\cdot 7)^3 = 0\cdot 343.$$

(b) This is an example of a *hypergeometric distribution* with parameters $N = 10$ and $n = 7$ but we can find the probability of the give event without reference to this general description, for it is:

$$\frac{\binom{7}{3}}{\binom{10}{3}} = \frac{7!}{4!3!}\cdot\frac{7!3!}{10!} = \frac{7\cdot 6\cdot 5}{10\cdot 9\cdot 8} = \frac{7}{24} = 0\cdot 292.$$

4. We want $P(X \leq 2)$. In general, $P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}$ and so we here need:

$$= e^{-\lambda}(1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!}) = e^{-4}(1 + 4 + 8) = \frac{13}{e^4} = 0\cdot 238 \text{ (3 d.p.)}.$$

5. Such processes follow a Poisson distribution with the mean number λ of misprints per page equal to $\frac{750}{500} = \frac{3}{2}$. The mean number per two pages (whether

or not they are consecutive) is $2 \cdot \frac{3}{2} = 3$. Hence we require $p = P(X = 0)$ where $X \sim \text{Po}(3)$, so that

$$p = e^{-\lambda}\frac{\lambda^0}{0!} = e^{-\lambda} = e^{-3} = 0 \cdot 0497 \text{ (3 s.f.)}.$$

6. The probability will the be the following ratio, the numerator of which is the number of ways of choosing exactly k blue marbles and $m-k$ red ones from the bag while the denominator is the total number of groups of marbles of size m that can be chosen:

$$\frac{\binom{n}{k}\binom{N-n}{m-k}}{\binom{N}{m}}.$$

7. Since there are N balls, the number of subsets of m balls that can be drawn from the bag is $\binom{N}{m}$. This is also equal to the sum, as k ranges from 0 to m of the number of ways of choosing k blue balls and $m-k$ red balls. Hence

$$\binom{N}{m} = \sum_{k=0}^{m}\binom{n}{k}\binom{N-n}{m-k} \qquad (2)$$

$$\Rightarrow \sum_{k=0}^{m}\frac{\binom{n}{k}\binom{N-n}{m-k}}{\binom{N}{m}} = 1.$$

Comment The identity (2) is often known as *Vandermonde's identity* (1772) but this fact was known to the Chinese mathematician Zhu Shijie (1303).

8. Here we have a Binomial distribution with $n = 12$ and $p = 0{\cdot}8$. Now $\lfloor(n+1|)p\rfloor = \lfloor 10 \cdot 4\rfloor = 10$, which is then the *mode* (outcome of highest probability) for this distribution. (The probability that $X = 10$ can be calculated as $0{\cdot}2384$.)

9. We put $\lambda = np = 500 \times 0 \cdot 002 = 1$ and approximate the distribution to that of $X \sim$Po(1). We want

$$P(X = 2) = e^{-1}\frac{1^2}{2!} = \frac{1}{2e} = 0 \cdot 184 \text{ 3 d.p.}$$

Comment This agrees, to 3 d.p. with the exact answer given through the binomial distribution.

10. Here we have a binomial distribution with $n = 90$ and $p = \frac{1}{36}$. We approximate this by $X \sim$Po(λ) where $\lambda = np = \frac{90}{36} = 2 \cdot 5$. Then

$$P(X \geq 2) = 1 - P(X = 0) - P(X = 1)$$

$$= 1 - e^{-2 \cdot 5}(1 + 2 \cdot 5) = 1 - \frac{3 \cdot 5}{e^{2 \cdot 5}} = 0 \cdot 713 \text{ 3 d.p.}$$

The probability that at least two double sixes are observed in 90 tosses of the dice is $0 \cdot 713$.

Problem Set 8

1. Let Y be the number of events in t time units. Then $Y \sim \text{Po}(\lambda t)$ and $P(Y=0) = e^{-\lambda t}$. Hence $P(\text{waiting time till first event} < t) = 1 - P(Y=0) = 1 - e^{-\lambda t}$, which is to say

$$F(t) = 1 - e^{-\lambda t} \Rightarrow f(t) = F'(t) = \lambda e^{-\lambda t} \; (t \geq 0).$$

2. Here $\lambda = 3$ so that $f(t) = 3e^{-3t}$ $(t \geq 0)$. Hence

$$P(T > 1) = \int_1^\infty 3e^{-3t}\, dt = [-e^{-3t}]_{t=1}^{\infty} = 0 - (-e^{-3}) = e^{-3} = 0 \cdot 050 \text{ 3 d.p.}$$

3.

$$E(Z) = E(\frac{X-\mu}{\sigma}) = \int_{-\infty}^{\infty} \frac{x-\mu}{\sigma} f(x)\, dx$$

$$= \frac{1}{\sigma}(\int_{-\infty}^{\infty} x f(x)\, dx - \frac{\mu}{\sigma}\int_{-\infty}^{\infty} f(x)\, dx = \frac{\mu}{\sigma} - \frac{\mu}{\sigma} = 0.$$

Hence $E(Z) = 0$. It follows that the variance $Var(Z)$ is given by

$$E(Z^2) = \int_{-\infty}^{\infty} (\frac{X-\mu}{\sigma})^2 f(x)\, dx = \frac{1}{\sigma^2}\int_{-\infty}^{\infty} (X-\mu)^2 f(x)\, dx$$

$$= \frac{1}{\sigma^2} \cdot \sigma^2 = 1.$$

4. We have $X = \mu + \sigma Z$ so that

$$P(X \leq x) = P(\mu + \sigma Z \leq x) = P(Z \leq \frac{x-\mu}{\sigma}) = F(\frac{x-\mu}{\sigma}).$$

Hence the pdf $g(x)$of X is given by

$$g(x) = (F(\frac{x-\mu}{\sigma}))' = \frac{1}{\sigma} f(\frac{x-\mu}{\sigma}).$$

5. In the case where $Z = N(0,1)$ we have the pdf $g(x)$ of $X = \mu + \sigma Z$ is given by

$$\frac{1}{\sigma}\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{2}\right)^2} = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\cdot\left(\frac{x-\mu}{\sigma}\right)^2}.$$

6. Since $f(x)$ is an even function, the graph of $f(x)$ is symmetric with respect to the y-axis (it is the standard *bell curve*). It follows that $P(X \geq x) = P(X \leq x)$ and so

$$\Phi(-x) = P(X \geq x) = 1 - P(X \leq x) = 1 - \Phi(x), \; (-\infty < x < \infty).$$

Or equivalently

$$\Phi(x) + \Phi(-x) = 1 \; \forall x.$$

7.

$$P(|X-50|<\sqrt{8}) = P(-\sqrt{8}<X-50<\sqrt{8})$$

$$P(-1<\frac{X-50}{\sqrt{8}}<1) = P(-1<Z<1)$$

$$= \Phi(1)-\Phi(-1) = \Phi(1)-(1-\Phi(1))$$

$$= 2\Phi(1)-1 = 2(0\cdot 8413)-1 = 0\cdot 6826.$$

Therefore $P(|x-50|<\sqrt{8)}=0\cdot 6826$.

8. We match the mean and variance of the two distributions. Here $\mu = np = 400\times 0\cdot 35 = 140$ and $\sigma = npq = 140\times 0\cdot 65 = 91$. Hence our approximating normal distrtibution is $X\sim N(140,91)$. Then

$$P(120\leq X\leq 150)\to P(119.5\leq X\leq 150.5)\text{ (continuity correction)}$$

$$= P(\frac{119\cdot 5-140}{\sqrt{91}}\leq\frac{X-140}{\sqrt{91}}\leq\frac{150\cdot 5-140}{\sqrt{91}})$$

$$= P(-2\cdot 149\leq Z\leq 1\cdot 101) = \Phi(1\cdot 101)-\Phi(-2\cdot 149) = \Phi(1\cdot 101)-(1-\Phi(2\cdot 149))$$

$$\Phi(1\cdot 101)+\Phi(2\cdot 149)-1 = 0\cdot 8465+0\cdot 9842-1 = 0\cdot 8307.$$

The probability that between 120 and 150 brown-eyed people in the sample is $0\cdot 8307$.

Comment The rule of thumb is that the normal with matching mean and variance is a good approximation to the binomial for large n and when p is close to $\frac{1}{2}$, the latter guaranteeing that the shape of the distribution is not too skewed to one end but is more like the normal bell-shape.

9. We approximate the underlying Poisson distribution with mean (and variance) $\lambda = 25$ by a normal random variable X with the same mean and variance, $X\sim N(25,25)$. Again, using a continuity correction in order to reduce rounding error we cacluate

$$P(22\cdot 5\leq X\leq 27\cdot 5) = P(\frac{22\cdot 5-25}{5}\leq\frac{X-25}{5}\leq\frac{27\cdot 5-25}{5})$$

$$= P(-0\cdot 5\leq Z\leq 0\cdot 5) = 2\Phi(0\cdot 5)-1 = 2(0\cdot 6915)-1 = 0\cdot 383.$$

Therefore the probability that between 23 and 27 particles are detected in any give second is $0\cdot 383$.

Comment The heuristic for this approximation to be sound is that $\lambda>20$.

10. We are given $X\sim N(\mu,(1\cdot 3)^2)$. The 95% confidence interval for μ is $\overline{x}\pm 1\cdot 96\frac{\sigma}{\sqrt{n}}$ so the width of the interval is $2\times 1\cdot 96\frac{\sigma}{\sqrt{n}}$. In this case we therefore need the least positive integer n such that

$$2\times 1\cdot 96\times\frac{1\cdot 3}{\sqrt{n}}<2\Rightarrow\sqrt{n}>1\cdot 96\times 1\cdot 3 = 2\cdot 548$$

$$\Rightarrow n>6\cdot 49,$$

so that $n = 7$ tests are needed.

Problem Set 9

1. We have $E((X-a)^2) = E(X^2) - 2a\mu + a^2$. Since $E(X^2)$ is a constant, we seek the value of a that minimizes $a^2 - 2a\mu = a(a - 2\mu)$. This is a parabola in a with roots 0 and 2μ and so the minimum occurs at $a = \mu$.

2.

$$\mathbb{E}(X) = \sum_{k=1}^{n} k \binom{n}{k} p^k q^{n-k} = \sum_{k=1}^{n} \frac{n!}{(k-1)!(n-k)!} p^k q^{n-k} =$$

$$np \sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!((n-1)-(k-1))!} p^{k-1} q^{(n-1)-(k-1)} =$$

$$= np \sum_{k=0}^{n-1} \binom{n-1}{k-1} p^k q^{(n-1)-k} = np(p+q)^{n-1} = np.$$

3.

$$\mathbb{E}(X(X-1)) = \sum_{k=2}^{n} k(k-1) \binom{n}{k} p^k q^{n-k} = \sum_{k=2}^{n} \frac{n!}{(k-2)!(n-k)!} p^k q^{n-k}$$

$$= n(n-1)p^2 \sum_{k=2}^{n} \frac{(n-2)!}{(k-2)!((n-k)!} p^{k-2} q^{(n-2)-(k-2)} =$$

$$n(n-1)p^2 \sum_{k=0}^{n-2} \frac{(n-2)!}{k!((n-2)-k)!} p^k q^{(n-2)-k} = n(n-1)p^2(p+q)^{n-2} = n(n-1)p^2.$$

4. Now in general

$$Var(X) = \mathbb{E}(X^2) - \mathbb{E}^2(X) = \mathbb{E}(X(X-1)) + \mathbb{E}(X) - \mathbb{E}^2(X).$$

In this case we get from the results of the two previous questions:

$$Var(X) = n(n-1)p^2 + np - n^2p^2 = np - np^2 = np(1-p) = npq.$$

Hence $\sigma(X) = \sqrt{npq}$.

5.

$$\mathbb{E}(X) = \sum_{k=1}^{\infty} k \frac{\lambda^k e^{-\lambda}}{k!} = \lambda e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} =$$

$$\lambda e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = \lambda e^{-\lambda} e^{\lambda} = \lambda.$$

6.
$$\mathbb{E}(X(X-1)) = \sum_{k=2}^{\infty} k(k-1)\frac{\lambda^k e^{-\lambda}}{k!} = \lambda^2 e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^{k-2}}{(k-2)!} =$$
$$= \lambda^2 e^{-\lambda} e^{\lambda} = \lambda^2.$$

7. Again $Var(X) = \mathbb{E}(X(X-1)) + \mathbb{E}(X) - \mathbb{E}(X) = \lambda^2 + \lambda - \lambda^2 = \lambda$; $\sigma(X) = \sqrt{\lambda}$.

8. $\int_0^\infty \lambda e^{-\lambda x} dx = \lambda\big[-\frac{1}{\lambda}e^{-\lambda x}\big]_0^\infty = -\big[0-1\big] = 1$ and since $f(x) \geq 0 \; \forall x$, it follows that $f(x)$ is a pdf. Next

$$F(x) = \int_0^x f(t)\,dt = -\big[e^{-\lambda t}\big]_0^{t=x} = -\big[e^{-\lambda x} - 1\big] = 1 - e^{-\lambda x} \; (x \geq 0).$$

Comment We use the symbol t in the integral so that the symbol x does not simultaneously stand for a fixed value and also for the variable of integration. Since the value of the integral is independent of the symbol used as the variable of integration, the variable t is sometimes referred to as a *dummy variable*, meaning that it has no particular meaning in itself, and so could be any symbol that is not used elsewhere in the calculation.

9. First consider the integral $I = \int xe^{-\frac{1}{2}x^2} dx$. Put $u = -\frac{1}{2}x^2 \Rightarrow du = -x\,dx$. Hence $I = -\int e^u du = -e^{-\frac{1}{2}x^2} + c$. Hence

$$\mathbb{E}(X) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} xe^{-\frac{1}{2}x^2} dx = -\frac{1}{\sqrt{2\pi}}\big[e^{-\frac{1}{2}x^2}\big]_{-\infty}^{\infty} = -\frac{1}{\sqrt{2\pi}}\big[0-0\big] = 0.$$

Comment That the mean is 0 also follows from the fact that the pdf is an even function so the integrand of $\mathbb{E}(X)$ is odd. A similar comment applies to the latter calculation in the following question.

10. The variance of X is $\mathbb{E}(X-\mu)^2 = \mathbb{E}(X^2)$. Consider the integral $I = \int x^2 e^{-\frac{1}{2}x^2} dx$. Integrate by parts, putting $u = x$, $dv = xe^{-\frac{1}{2}x^2} dx$ so that $du = dx$ and $v = -e^{-\frac{1}{2}x^2}$. Then

$$I = xe^{-\frac{1}{2}x^2} + \int e^{-\frac{1}{2}x^2} dx.$$

Hence
$$\mathbb{E}(X^2) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} xe^{-\frac{1}{2}x^2} dx + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2} dx.$$

Now the second integral in the previous line is equal to 1 as it is that of a pdf, while for the first we get

$$-\frac{1}{\sqrt{2\pi}}\big[e^{-\frac{1}{2}x^2}\big]_{\infty}^{\infty} = -\frac{1}{\sqrt{2\pi}}\big[0-0\big] = 0.$$

It follows that $\sigma^2(X) = \sigma(X) = \mathbb{E}(X^2) = 1$.

Problem Set 10

1. Our distribution is $P(X=k)=\binom{n}{k}p^kq^{n-k}$ $(q=1-p,\ 0\le k\le n)$. Hence

$$\mathbb{E}(e^{tX})=\sum_{k=0}^{n}e^{tk}\binom{n}{k}p^kq^{n-k}=\sum_{k=0}^{n}\binom{n}{k}(pe^t)^kq^{n-k}=(pe^t+q)^n;$$

$$\therefore M(t)=(p(e^t-1)+1)^n.$$

2. Continuing with the solution to Q1 we obtain:

$$\mathbb{E}'(e^{tX})=npe^t(pe^t+q)^{n-1}\Rightarrow \mathbb{E}''(e^{tX})=np(pe^t+q)^{n-1}+n(n-1)p^2e^{2t}(pe^t+q)^{n-2}.$$

Hence

$$\mathbb{E}(X)=M'(0)=npe^0(pe^0+q)^{n-1}=np.$$

$\mathbb{E}(X^2)=M^{(2)}(0)=np(pe^0+q)^{n-1}+n(n-1)p^2(pe^0+q)^{n-2}=np+n(n-1)p^2$ Hence $\sigma^2=\mathbb{E}(X^2)-\mathbb{E}^2(X)=np+n(n-1)p^2-n^2p^2=np-np^2=np(1-p)=npq.$

3.

$$\mathbb{E}(e^{tX})=\int_a^b\frac{e^{tx}}{b-a}dx=\frac{1}{t(b-a)}[e^{tX}]_a^b=$$

$$\frac{1}{t(b-a)}\left(e^{tb}-e^{ta}\right),\ t\neq 0;\mathbb{E}(e^0)=1.$$

4.

$$M'(t)=-\frac{1}{t^2(b-a)}\left(e^{tb}-e^{ta}\right)+\frac{1}{t(b-a)}\left(be^{tb}-ae^{ta}\right)\Rightarrow$$

$$(b-a)M'(t)=\frac{e^{bt}(bt-1)-e^{at}(at-1)}{t^2}.$$

Taking the limit as $t\to 0$ we obtain:

$$(b-a)M'(0)=\lim_{t\to 0}\frac{e^{bt}(bt-1)-e^{at}(at-1)}{t^2}=$$

$$\lim_{t\to 0}\frac{b^2te^{bt}-a^2te^{at}}{2t}=\lim_{t\to}\frac{b^2e^{bt}+b^3te^{bt}-a^2e^{at}-a^3te^{at}}{2}=\frac{b^2-a^2}{2}\Rightarrow$$

$$M'(0)=\frac{b^2-a^2}{2(b-a)}=\frac{(b-a)(b+a)}{2(b-a)}=\frac{b+a}{2}.$$

Comment In this case the mgf method involves a lot more work that a direct calculation. Indeed from symmetry it is obvious that $\mu=\frac{b+a}{2}$ for the uniform distribution. Direct calculation also gives $\sigma^2=\frac{(b-a)^2}{12}$.

5.

$$\mathbb{E}(e^{tX})=\sum_{k=0}^{\infty}e^{tk}\frac{\lambda^ke^{-\lambda}}{k1}=e^{-\lambda}\sum_{k=0}^{\infty}\frac{(\lambda e^t)^k}{k!}=e^{-\lambda}e^{\lambda e^t}\Rightarrow$$

$$M(t) = e^{\lambda(e^t-1)}.$$

6. Continuing from Q5 we have

$$M'(t) = \lambda e^t \cdot e^{\lambda(e^t-1)} = \lambda e^{\lambda e^t - \lambda + t} \Rightarrow M'(0) = \lambda e^{\lambda-\lambda+0} = \lambda e^0 = \lambda.$$

$$M''(t) = \lambda(\lambda e^t + 1)e^{\lambda e^t + t - \lambda} \Rightarrow M''(0) = \lambda(\lambda+1)e^{\lambda+0-\lambda} = \lambda(\lambda+1);$$

Hence

$$\sigma^2 = \mathbb{E}(X^2) - \mathbb{E}^2(X) = \lambda(\lambda+1) - \lambda^2 = \lambda^2 + \lambda - \lambda^2 = \lambda.$$

7. We first find

$$\mathbb{E}(X(X-1)(X-2)) = \sum_{k=2}^{\infty} \frac{k(k-1)(k-2)\lambda^k e^{-\lambda}}{k!} = e^{-\lambda}\lambda^3 \sum_{k=3}^{\infty} \frac{\lambda^{k-3}}{(k-3)!}$$

$$= e^{-\lambda}\lambda^3 \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = \lambda^3 e^{-\lambda} e^{\lambda} = \lambda^3. \text{Now}$$

$$\mathbb{E}(X^3) = \mathbb{E}((X(X-1)(X-2)) + 3\mathbb{E}(X^2) - 2\mathbb{E}(X) = \lambda^3 + 3\lambda(\lambda+1) - 2\lambda = \lambda^3 + 3\lambda^2 + \lambda.$$

Hence

$$\mathbb{E}\big(\frac{X-\mu}{\sigma}\big)^3 = \frac{1}{\lambda^{3/2}}\big(\mathbb{E}(X^3) - 3\lambda\mathbb{E}(X^2) + 3\lambda^2\mathbb{E}(X) - \lambda^3\big)$$

$$= \frac{1}{\lambda^{3/2}}\Big(\lambda^3 + 3\lambda^2 + \lambda - 3\lambda^3 - 3\lambda^2 + 3\lambda^3 - \lambda^3\Big) = \frac{\lambda}{\lambda^{3/2}} = \frac{1}{\sqrt{\lambda}}.$$

8.

$$\mu = \sum_{k=0}^{\infty} kP(X=k) \geq \sum_{k=a}^{\infty} kP(X=k) \geq a\sum_{k=a}^{\infty} P(X=k) = aP(X \geq k)$$

$$\Rightarrow P(X \geq k) \leq \frac{\mu}{a}.$$

9. The random variable $Y = (X-\mu)^2$ is a discrete random variable on the non-negative integers with mean σ^2 so applying the Markov Inequality to Y we obtain:

$$P\big((X-\mu)^2 \geq k^2\sigma^2\big) \leq \frac{\sigma^2}{k^2\sigma^2} = \frac{1}{k^2};$$

Now $(X-u)^2 \geq k^2\sigma^2 \Leftrightarrow |X-\mu| \geq k\sigma$, which gives the required conclusion:

$$P\big(|X-\mu| \geq k\sigma\big) \leq \frac{1}{k^2}.$$

10. We have $X \sim B(200,\, 0\cdot 5)$ so that $\mu = np = 100$ and $\sigma^2 = npq = 50$. Hence applying the Chebyshev Inequality we obtain:

$$P(|X-100| \geq 2\sqrt{50}) \leq \frac{1}{2^2} \Rightarrow P(|X-100| \geq 10\sqrt{2}) \leq \frac{1}{4}.$$