Mathematics 201 Calculus of Several Variables Solutions

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June 27, 2017

Solutions and Comments for the Problems

Problem Set 1

1.
$$I = \int_{y=1}^{2}\int_{x=0}^{1}(x^2y+xy^2)\,dxdy = \int_{y=1}^{2}[\frac{x^3y}{3}+\frac{x^2y^2}{2}]_{x=0}^{1}\,dy$$
$$= \int_{y=1}^{2}\big((\frac{y}{3}+\frac{y^2}{2})-(0+0)\big)\,dy = [\frac{y^2}{6}+\frac{y^3}{6}]_1^2 = \frac{1}{6}[(4+8)-(1+1)]$$
$$= \frac{1}{6}(12-2) = \frac{10}{6} = \frac{5}{3}.$$

2.
$$I = \int_0^{\frac{\pi}{4}}\int_0^1 2xy\sin y\,dxdy = \int_0^{\frac{\pi}{4}} y\sin y\big(\int_0^1 2x\,dx\big)\,dy$$
$$= \int_0^{\frac{\pi}{4}} y\sin y[x^2]_{x=0}^{x=1}\,dy = \int_0^{\frac{\pi}{4}} y\sin y\,dy.$$

Integrating by parts with $u=y$, $dv=\sin y\,dy$ gives $du=dy$ and $v=-\cos y$:

$$I = [-y\cos y]_0^{\frac{\pi}{4}} + \int_0^{\frac{\pi}{4}}\cos y\,dy = [-y\cos y+\sin y]_0^{\frac{\pi}{4}}$$
$$= [(-\frac{\pi}{4}\cdot\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2})-(-0+0)] = \frac{\sqrt{2}}{8}(4-\pi).$$

3. We change the order of integration because we do not know how to integrate e^{y^3} (indeed no elementary function differentiates to gives this). For a fixed value of y, x varies from a lower value of 0 up to an upper limit of y^2 (for a given value of y, the point on the boundary curve (x,y) satisfies $y=\sqrt{x}$ so that $x=y^2$). The range of y is then from 0 up to 1 Hence the integral is re-written as

$$I = \int_{y=0}^{1}\int_{x=0}^{y^2} e^{y^3}\,dxdy = \int_{y=0}^{1}[e^{y^3}x]_0^{x=y^2}\,dy$$
$$= \int_{y=0}^{1}(y^2e^{y^3}-0)\,dy = [\frac{e^{y^3}}{3}]_{y=0}^{1}$$
$$= \frac{e}{3}-\frac{1}{3} = \frac{e-1}{3}.$$

4. The region of integration is a right-triangle bounded by the lines $y=x$, $y=\frac{\pi}{2}$ and the y-axis. The given limits have inner variable y. To reverse the

order of integration we use horizontal stripes. The limits in this order are (inner) x from 0 to y; (outer) y from 0 to $\frac{\pi}{2}$. Hence the integral becomes

$$\int_0^{\frac{\pi}{2}} \int_0^y \frac{\sin y}{y}\, dxdxy = \int_0^{\frac{\pi}{2}} \frac{\sin y}{y} \Big(\int_0^y dx \Big) dy$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin y}{y} (y-0)\, dy = \int_0^{\frac{\pi}{2}} \sin y\, dy$$

$$= -[\cos y]_0^{\frac{\pi}{2}} = -[0-1] = 1.$$

5. The region of integration is the right angled triangle with vertices $(0,0)$, $(1,0)$, $(1,2)$, the sides being the lines $y=0$, $x=1$ and $y=2x$. For a fixed value of x, y ranges over the interval $[0,2x]$ as determining the inner limits of the transformed integral. The alternative form integral is then:

$$\int_0^1 \int_0^{y=2x} e^{x^2} dydy = \int_0^1 e^{x^2} \Big(\int_0^{2x} dy \Big) dx = \int_0^1 e^{x^2} \left[y \right]_0^{2x} dx =$$

$$= \int_0^1 2xe^{x^2} dx = [e^{x^2}]_0^1 = (e^{1^2} - e^0) = e - 1.$$

6.

$$I = \int_0^{x=1} \int_0^{y=1} [\frac{1}{2} z^2 x e^{xy}]_0^{z=2}\, dydx$$

$$= \int_0^{x=1} \int_0^{y=1} (2xe^{xy} - 0)\, dydx = 2 \int_0^{x=1} \int_0^{y=1} xe^{xy}\, dydx$$

$$= 2 \int_0^{x=1} [e^{xy}]_0^{y=1}\, dx = 2 \int_0^{x=1} (e^x - e^0)\, dx$$

$$= 2[e^x - x]_0^1 = 2[(e-1) - (e^0 - 0)] = 2(e-2).$$

7.

$$I = \int_0^{\frac{\pi}{2}} \cos z \Big(\int_0^1 x\sqrt{1-x^2}\, dx \Big)\, dz.$$

For the integral in x put $u = 1 - x^2$ so that $du = -2xdx$ so that $x\,dx = -\frac{1}{2}\,du$; $x=1$ gives $u=0$ and $x=0$ gives $u=1$. Hence

$$\int_0^1 x\sqrt{1-x^2}\, dx = -\frac{1}{2} \int_1^0 u^{\frac{1}{2}}\, du = \frac{1}{2} [\frac{2}{3} u^{\frac{3}{2}}]_0^1 = \frac{1}{3}(1-0) = \frac{1}{3}.$$

Hence

$$I = \frac{1}{3} \int_0^{\frac{\pi}{2}} \cos z\, dz = \frac{1}{3} [\sin z]_0^{\frac{\pi}{2}} = \frac{1}{3}(1-0) = \frac{1}{3}.$$

8.

$$I = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} dxdxdz = \int_0^1 \int_0^{1-x} [z]_{z=0}^{z=1-x-y}\, dxdy$$

$$= \int_0^1 \int_0^{1-x} (1-x-y)\, dxdy = \int_0^1 [y - xy - \frac{1}{2}y^2]_{y=0}^{y=1-x}\, dx$$

$$= \int_0^1 \left(((1-x) - x(1-x) - \frac{1}{2}(1-x)^2\right) dx = \int_0^1 \frac{1}{2}(1-x)^2\, dx$$

$$= -\frac{1}{6}[(1-x)^3]_0^1 = -\frac{1}{6}[0-1] = \frac{1}{6}.$$

On the other hand, the volume is that of a tetrahedron with vertices $(1,0,0), (0,1,0), (0,0,1), (0,0,0)$. The base is an equilateral triangle T of side length $\sqrt{1^2 + (-1)^2 + 0^2} = \sqrt{2}$. The altitude h of T satisfies

$$(\frac{1}{\sqrt{2}})^2 + h^2 = (\sqrt{2})^2 \Rightarrow h^2 = 2 - \frac{1}{2} = \frac{3}{2} \Rightarrow h = \frac{\sqrt{6}}{2}.$$

Hence the area of T is $\frac{1}{2} \cdot \sqrt{2} \cdot \frac{\sqrt{6}}{2} = \sqrt{12} = 2\sqrt{3}$. The vector $\mathbf{v} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ is perpendicular to the plane and its direction is that of the line $x = y = z$, which meets the plane $x + y + z = 1$ at $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. Therefore the height of the tetrahedron is $\sqrt{3(\frac{1}{3})^2} = \frac{1}{\sqrt{3}}$. Therefore the volume of the tetrahedron is

$$V = \frac{1}{3}(2\sqrt{3})(\frac{1}{\sqrt{3}}) = \frac{1}{6}.$$

9. $z = \sqrt{4 - x^2}$, $\frac{\partial z}{\partial x} = -x(4 - x^2)^{-\frac{1}{2}}$, $\frac{\partial z}{\partial y} = 0$ and so

$$S = \int_0^4 \int_0^1 \sqrt{1 + \frac{x^2}{4 - x^2}} dxdy = 4 \int_0^1 \sqrt{\frac{4}{4 - x^2}} dx = 8 \int_0^1 \frac{dx}{\sqrt{4 - x^2}}$$

$$= 8[\arcsin \frac{x}{2}]_0^1 = 8[\arcsin \frac{1}{2} - \arcsin 0] = 8(\frac{\pi}{6} - 0) = \frac{4\pi}{3}.$$

10. The surfaces meet when $2z = 8 \Rightarrow z = 4$ and the region over which the integration takes place is the circle $R:\ x^2 + y^2 = 8 = (2\sqrt{2})^2$. We have $z = \frac{1}{2}(x^2 + y^2) \Rightarrow \frac{\partial z}{\partial x} = x$, $\frac{\partial z}{\partial y} = y$. Thus $S = \int\int_R \sqrt{1 + x^2 + y^2} dxdy$. Transforming to polar coordinates we get:

$$S = \int_0^{2\pi} \int_0^{2\sqrt{2}} r\sqrt{1 + r^2} drd\theta = 2\pi \int_0^{2\sqrt{2}} r\sqrt{1 + r^2} dr.$$

Substituting $u = 1 + r^2$ we get $\frac{1}{2}du = rdr$, $r = 0 \to u = 1$, $r = 2\sqrt{2} \to u = 9$, giving

$$S = \pi \int_1^9 u^{\frac{1}{2}} du = \frac{2\pi}{3}[u^{\frac{3}{2}}]_1^9 = \frac{2\pi}{3}(27 - 1) = \frac{52\pi}{3}.$$

Problem Set 2

1. We perform the calculation for cylindrical coordinates where the third variable z is unchanged by the transformation. We then get

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial y} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= r\cos^2\theta - (-r\sin^2\theta) = r(\cos^2\theta + \sin^2\theta) = r.$$

Hence $|J| = |r| = r$ as we take $r \geq 0$.

2. First leaf has range of θ of $0 \leq \theta \leq \frac{\pi}{3}$ and so our area A is given by:

$$A = 3\int_0^{\frac{\pi}{3}} \int_0^{r=\sin 3\theta} r dr d\theta = \frac{3}{2}\int_0^{\frac{\pi}{3}} [r^2]_0^{\sin 3\theta} d\theta$$

$$\Rightarrow A = \frac{3}{2}\int_0^{\frac{\pi}{3}} \sin^2 3\theta d\theta = \frac{3}{4}\int_0^{\frac{\pi}{3}} (1 - \cos 6\theta) d\theta = \frac{3}{4}[\theta - \frac{\sin 6\theta}{6}]_0^{\frac{\pi}{3}} =$$

$$\frac{3}{4}[\frac{\pi}{3} - (\frac{\sin 2\pi}{6} - 0)] = \frac{\pi}{4}.$$

3. In polars, the variable r ranges from 0 to $+\infty$ as the polar angle θ runs through a full sweep from 0 to 2π. Replacing $x^2 + y^2$ by r^2 and $dxdy$ by $rdrd\theta$ the integral I takes on the form:

$$\int_0^{2\pi} \int_0^r re^{-r^2} dr d\theta = -\frac{1}{2}\int_0^{2\pi} [e^{-r^2}]_0^{\infty} d\theta = -\frac{1}{2}\int_0^{2\pi} [0 - 1] d\theta = \frac{1}{2}(2\pi) = \pi.$$

4. Let $J = \int_{-\infty}^{\infty} e^{-x^2} dx$. Now the integral I of Question 3 is equal to

$$(\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2} dx)(\int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2} dy) = J^2$$

$$\Rightarrow \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2} dx = \sqrt{\pi}.$$

5. Consider $I = \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2} dx$. Put $x = \sqrt{2}u$ to get $dx = \sqrt{2}du$, $e^{-\frac{1}{2}x^2} = e^{-\frac{1}{2}(\sqrt{2}u)^2} = e^{-\frac{1}{2}u^2}$; the limits of $\pm\infty$ remain the same when passing to the transformed variable u and so we obtain from the result of Question 4 that

$$I = \sqrt{2}\int_{-\infty}^{\infty} e^{-\frac{1}{2}u^2} du = \sqrt{2} \cdot \sqrt{\pi} = \sqrt{2\pi}.$$

Therefore we gain the required conclusion, that being:

$$\frac{1}{\sqrt{2\pi}}\int_0^{\infty} e^{-\frac{1}{2}x^2} dx = 1.$$

6.

$$y = \sqrt{1-(x-1)^2} \Rightarrow y^2 + (x-1)^2 = 1;$$

which is a semicircle, centred at $(1,0)$ of radius 1. We have $y^2 + x^2 - 2x + 1 = 1$ so that

$$x^2 + y^2 = 2x \Rightarrow r^2 = 2r\cos\theta \Rightarrow r = 2\cos\theta\ (0 \le \theta \le \frac{\pi}{2}.$$

Hence our integral I becomes

$$I = \int_0^{\frac{\pi}{2}} \int_0^{2\cos\theta} \frac{r(\cos\theta + \sin\theta)}{r^2} \cdot r dr d\theta$$

$$\int_0^{\frac{\pi}{2}} 2\cos\theta(\cos\theta + \sin\theta)\, d\theta = \int_0^{\frac{\pi}{2}} (2\cos^2\theta + 2\sin\theta\cos\theta)\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} ((1+\cos 2\theta) + \sin 2\theta)\, d\theta = [\theta + \frac{1}{2}\sin 2\theta - \frac{1}{2}\cos 2\theta]_0^{\frac{\pi}{2}}$$

$$= (\frac{\pi}{2} + 0 - \frac{1}{2}(-1)) - (0 + 0 - \frac{1}{2}) = \frac{\pi+2}{2}.$$

7. The area is equal to the value of I where

$$I = \int_0^{\theta=2\pi} \int_0^{r=1+\sin\theta} r dr d\theta = \frac{1}{2}\int_0^{\theta=2\pi} [r^2]_0^{1+\sin\theta}\, d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} (1+\sin\theta)^2\, d\theta = \frac{1}{2}\int_0^{2\pi} (1 + 2\sin\theta + \sin^2\theta)\, d\theta;$$

the first term in this integral is $\frac{2\pi}{2} = \pi$ while the second is 0 as $\sin\theta$ is periodic with period 2π. The third term is

$$\frac{1}{4}\int_0^{2\pi} (1-\cos 2\theta)\, d\theta = \frac{2\pi}{4}$$

as the second term in this integrand has period π and so also evaluates to 0. Overall then we obtain $I = \pi + \frac{\pi}{2} = \frac{3\pi}{2}$.

8. We have $z = 4 - r^2$ and $4 - r^2 = 0$ when $r = 2$ so our volume integral is:

$$V = \int_0^{2\pi} \int_{r=0}^{2} \int_{z=0}^{4-r^2} r\, dz dr d\theta = \int_0^{2\pi} \int_0^{2} (4-r^2) r\, dr d\theta$$

$$= \int_0^{2\pi} [2r^2 - \frac{r^4}{4}]_{r=0}^{2}\, d\theta = 2\pi(8-4) = 8\pi.$$

9.

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta\sin\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\sin\theta\cos\phi & r\cos\theta\sin\phi \\ \cos\phi & -r\sin\phi & 0 \end{vmatrix}$$

using third row expansion and repeatedly using $\cos^2 + \sin^2 = 1$ we get

$$J = \cos\phi(r^2\cos^2\theta\cos\phi\sin\phi + r^2\sin^2\theta\sin\phi\cos\phi) + r\sin\phi(r\cos^2\theta\sin^2\phi + r\sin^2\theta\sin^2\phi)$$

$$= r^2\cos^2\phi\sin\phi + r^2\sin^3\phi = r^2\sin\phi$$

and so $|J| = r^2\sin\phi$ as $\sin\phi \geq 0$ for all $0 \leq \phi \leq \pi$.

Comment Note that exchanging the order of the variables will swap rows or columns in the determinant. This may change the sign of J but leaves $|J|$ unaltered.

10.

$$M = \int_0^\pi \int_0^{2\pi} \int_0^a \rho|J|\, dr d\theta d\phi = \int_0^\pi \int_0^{2\pi} \int_0^a \rho_0(1 - \frac{r}{a})r^2 \sin\phi\, dr d\theta d\phi$$

$$\rho_0 \int_0^\pi \Big(\int_0^{2\pi} \Big(\int_0^a (r^2 - \frac{r^3}{a})\, dr\Big) d\theta \sin\phi\Big)\, d\phi = \rho_0 \int_0^\pi \Big(\int_0^{2\pi} [\frac{r^3}{3} - \frac{r^4}{4a}]_0^{r=a}\Big)\, d\theta\Big) \sin\phi d\phi$$

$$\rho_0 \int_0^\pi \Big(\int_0^{2\pi} (\frac{a^3}{3} - \frac{a^4}{4a} - 0)\, d\theta\Big) \sin\phi\, d\phi = \frac{\rho_0 a^3}{12} \int_0^\pi \Big(\int_0^{2\pi} d\theta\Big) \sin\phi\, d\phi$$

$$= \frac{\rho_0 a^3}{12} \cdot 2\pi \int_0^\pi \sin\phi\, d\phi = \rho_0 a^3 \frac{\pi}{6}[-\cos\phi]_0^\pi = \frac{\pi}{6}\rho_0 a^3[-(-1) - (-1)]$$

$$\therefore M = \frac{\pi}{3}\rho_0 a^3.$$

Comment Cylindrical coordinates are useful in problems with cylindrical symmetry such as cylinders and cones while spherical coordinates work well with spheres. The difficulty with multiple integrals is often in determining the limits of integration for the inner integrals. For example, in a triple integral the innermost integral has limits that in general will be functions of the two remaining variables. Cylindrical and spherical coordinates, applied to appropriate problems, leave you with limits that are simply constants.

Problem Set 3

1. Differentiating with respect to time we get

$$\tan(\phi(t)) = \frac{\dot{y}(t)}{\dot{x}(t)} \Rightarrow \sec^2(\phi(t))\frac{d\phi}{dt} = \frac{\dot{x}(t)\ddot{y}(t) - \dot{y}(t)\ddot{x}(t)}{\dot{x}^2};$$

writing $\sec^2(\phi(t)) = 1 + \tan^2(\phi(t)) = 1 + \frac{\dot{y}^2}{\dot{x}^2}$ and solving for $\frac{d\phi}{dt}$ (while surpressing the t in the notation) now gives

$$\frac{d\phi}{dt} = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}^2 + \dot{y}^2}.$$

2.

$$\kappa = \frac{d\phi}{ds} = \frac{d\phi}{dt} \cdot \frac{dt}{ds} = \left(\frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}^2 + \dot{y}^2}\right)\left(\frac{1}{\sqrt{\dot{x}^2 + \dot{y}^2}}\right)$$

$$\therefore \kappa = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{(\dot{x}^2 + \dot{y}^2)^{\frac{3}{2}}}.$$

3. $x = a\cos t$, $y = a\sin t$, $(0 \leq t \leq 2\pi)$. Hence $\dot{x} = -a\sin t$, $\dot{y} = a\cos t$, $\ddot{x} = -a\cos t$, $\ddot{y} = -a\sin t$. This gives:

$$\kappa = \frac{a^2\sin^2 t + a^2\cos^2 t}{(a^2\sin^2 t + a^2\cos^2 t)^{3/2}} = \frac{a^2(\sin^2 t + \cos^2 t)}{(a^2(\sin^2 t + \cos^2 t))^{3/2}} = \frac{a^2}{(a^2)^{3/2}} = \frac{a^2}{a^3} = \frac{1}{a}.$$

Therefore the radius of curvature is $\rho = \frac{1}{\kappa} = a$, the radius of the circle.

4. Here $x = t$, $\dot{x} = 1$, $\ddot{x} = 0$, $y = t^2$, $\dot{y} = 2t$, $\ddot{y} = 2$. Hence

$$\kappa = \frac{2 - 0}{(1^2 + (2t)^2)^{3/2}} = \frac{2}{(1 + 4x^2)^{3/2}}.$$

5. $y = \ln(\cos x)$, $(-\frac{\pi}{2} \leq x \leq \frac{\pi}{2})$. Again we use $x = t$, $y = \ln(\cos t)$ so that $\dot{x} = 1$, $\ddot{x} = 0$, $\dot{y} = -\frac{\sin t}{\cos t} = -\tan t$, $\ddot{y} = -\sec^2 t$. Hence

$$\kappa = -\frac{\sec^2 t}{(1^2 + \tan^2 t)^{3/2}} = -\frac{\sec^2 t}{(\sec^2 t)^{3/2}} = -\frac{1}{\sec t} = -\cos x.$$

6. We have

$$x = a(t - \sin t),\ \dot{x} = a(1 - \cos t),\ y = a(1 - \cos t),\ \dot{y} = a\sin t. \text{Hence}$$

$$L = \int_0^\alpha \sqrt{\dot{x}^2 + \dot{y}^2}dt = \int_0^\alpha \sqrt{a^2(1 + \cos^2 t - 2\cos t + \sin^2 t)}dt$$

$$= a\sqrt{2}\int_0^\alpha \sqrt{1 - \cos t}dt.$$

Now $1 - \cos t = 2\sin^2\frac{t}{2}$ and so

$$L = 2a\int_0^\alpha \sin\frac{t}{2}dt = -4a[\cos\frac{t}{2}]_0^\alpha = -4a[\cos\frac{\alpha}{2} - 1]$$

$$\Rightarrow L = 4a(1 - \cos\frac{\alpha}{2}) = 8a\sin^2\frac{\alpha}{4}.$$

Putting $\alpha = 2\pi$ we get the length of one full arch of the cycloid is $L = 8a$.

7. We also need $\ddot{x} = a\sin t$, $\ddot{y} = a\cos t$. Hence

$$\rho = \frac{(\dot{x}^2 + \dot{y}^2)^{3/2}}{\dot{x}\ddot{y} - \dot{y}\ddot{x}} = \frac{(a^2(1 + \cos^2 t - 2\cos t + \sin^2 t))^{3/2}}{a^2(\cos t - \cos^2 t - \sin^2 t)} =$$

$$= -a\frac{(2 - 2\cos t)^{3/2}}{(1 - \cos t)} = -2a\sqrt{2}\sqrt{1 - \cos t} = -4a|\sin\frac{t}{2}|.$$

8. The area A under one arch of the cycloid is given by:

$$\int_0^{2\pi} y\frac{dx}{dt}dt = \int_0^{2\pi} a(1-\cos t)a(1-\cos t)dt = a^2\int_0^{2\pi}(1-2\cos t+\cos^2 t)dt$$

$$= a^2(2\pi + \frac{1}{2}\int_0^{2\pi}(1+\cos 2t)dt = a^2(2\pi+\pi) = 3a^2\pi$$

Comment: It is worth remembering that the integral of a constant is just that constant times the length of the interval of integration and that the integral of $\sin nx$ or $\cos nx$ $(n = 1, 2, \cdots)$ over an interval of length 2π is, by periodicity, always 0.

9. By the Fundamental theorem of calculus we have $\dot{x}(t) = \cos(t^2)$, $\dot{y(t)} = \sin(t^2)$ and so the required length L is:

$$L = \int_0^{t_0} \sqrt{\dot{x}(t)^2 + \dot{y}(t)^2}dt = \int_0^{t_0} \sqrt{\cos^2(t^2) + \sin^2(t^2)}dt$$

$$= \int_0^{t_0} dt = [t]_0^{t_0} = t_0 - 0 = t_0.$$

10.

$$\kappa = \frac{2t\cos(t^2)\cos(t^2) + 2t\sin(t^2)\sin(t^2)}{(\cos^2(t^2)+\sin^2(t^2))^{\frac{3}{2}}}$$

$$= \frac{2t(\cos^2(t^2)+\sin^2(t^2))}{1^{\frac{3}{2}}} = 2t\cdot\frac{1}{1} = 2t.$$

Hence $\frac{d\kappa}{dt} = 2$, is constant.

Comment Since distance equals time along this curve, a vehicle following this curve at constant speed has constant angular acceleration.

Problem Set 4

1. By definition, for some bound B,

$$f = O(1) \Leftrightarrow f(x) \leq K \cdot 1 = K \text{ for some } K \in \mathbb{R}.$$

On the other hand

$$f = o(1) \Leftrightarrow \lim_{x\to\infty}\frac{f(x)}{1} = \lim_{x\to\infty} f(x) = 0.$$

2.

$$\lim_{x\to\infty}\frac{x^m}{x^{m+1}} = \lim_{x\to\infty}\frac{1}{x} = 0$$

so that, by definition, $x^m = o(x^{m+1})$.

3. We apply L'Hopital's rule twice:

$$\lim_{x\to 0}\frac{\mathrm{cosec}x-\cot x}{x}=\lim_{x\to 0}\frac{1-\cos x}{x\sin x}=\lim_{x\to 0}\frac{\sin x}{\sin x+x\cos x}$$
$$=\lim_{x\to 0}\frac{\cos x}{2\cos x-x\sin x}=\frac{1}{2}.$$

In particular this shows that $\mathrm{cosec}x-\cot x\sim\frac{x}{2}$ as $x\to 0$.

4. Let $h=O(f)$ and $k=O(g)$ so that $h+k$ is a typical function in $O(f)+O(g)$. Then, by definition $|h(x)|\leq Kf(x)$ and $|k(x)|\leq Mg(x)$ say, and so

$$|(h+k)(x)|=|h(x)+k(x)|\leq|h(x)|+|k(x)|\leq Kf(x)+Mg(x).$$

Let $N=\max\{K,M\}$. Then we have

$$|(h+k)(x)|\leq N(f(x)+g(x))=N(f+g)(x)$$

so that $h+k=O(f+g)$.

Conversely let $h=O(f+g)$ so that

$$|h(x)|\leq K(f+g)(x)=K(f(x)+g(x))=Kf(x)+Kg(x)\leq 2\max\{Kf(x),Kg(x)\}$$

Hence $|h(x)|\leq 2Kf(x)$ and $|h(x)|\leq 2Kg(x)$ and so $h(x)=O(f(x))$ and $h(x)=O(g(x))$. It follows that $|\frac{1}{2}h(x)|\leq Kf(x)$ and $|\frac{1}{2}h(x)|\leq Kg(x)$ so that $h(x)=\frac{1}{2}h(x)+\frac{1}{2}h(x)=O(f)+O(g)$.

5. Continuing with the notation of Question 4, we have $hk=O(f)O(g)$ so that

$$|(hk)(x)|=|h(x)k(x)|=|h(x)|\cdot|k(x)|\leq Kf(x)\cdot Mg(x)$$
$$=KM\cdot f(x)g(x)=(KM)(fg(x))$$

so that $hk=O(fg)$.

Conversely suppose that $h=O(fg)$ so that for some constant K we have

$$|h(x)|\leq K(fg)(x)=Kf(x)\cdot g(x)$$

then

$$|\frac{h(x)}{g(x)}|\leq Kf(x),\ \ g(x)\leq 1\cdot g(x)$$

so that $h(x)=\frac{h(x)}{g(x)}\cdot g(x)$ with $\frac{h(x)}{g(x)}=O(f)$ and $g(x)=O(g)$ and therefore $h(x)=O(f)O(g)$.

6. Let $h=O(f)o(g)$ so we may write $h(x)=r(x)s(x)$ say with $|r(x)|\leq Kf(x)$ and $\frac{s(x)}{g(x)}\to 0$ (as $x\to\infty$ let us say). Then

$$\frac{h(x)}{(fg)(x)}\leq\frac{|h(x)|}{(fg)(x)}=\frac{|r(x)|\cdot|s(x)|}{f(x)g(x)}\leq\frac{Kf(x)|s(x)|}{f(x)g(x)}=\frac{K|s(x)|}{g(x)}\text{ hence}$$

$$|\lim_{x\to\infty}\frac{h(x)}{(fg)(x)}|\leq K\lim_{x\to\infty}\frac{|s(x)|}{g(x)}=K\lim_{x\to\infty}|\frac{s(x)}{g(x)}|$$

$$= K\Big|\lim_{x\to\infty}\frac{s(x)}{g(x)}\Big| = K\cdot 0 = 0$$

$$\therefore \lim_{x\to\infty}\frac{h(x)}{(fg)(x)} = 0$$

so that $h = o(fg)$, as required.

Conversely, suppose that $h = o(fg)$ so that

$$\lim_{x\to\infty}\frac{h(x)}{(fg)(x)} = \lim_{x\to\infty}\frac{h(x)}{f(x)g(x)} = 0$$

$$\Rightarrow \lim_{x\to\infty}\frac{(h(x)/f(x))}{g(x)} = 0,$$

and hence

$$h(x) = f(x)\cdot\frac{h(x)}{f(x)}$$

with $f = O(f)$ and $\frac{h}{f} = o(g)$, as required.

7. We are given that $f \sim g$, which is to say that $\lim_{x\to\infty}\frac{f(x)}{g(x)} = 1$. Take $h = o(g)$ so that $\lim_{x\to\infty}\frac{h(x)}{g(x)} = 0$. Then

$$\lim_{x\to\infty}\frac{f(x)+h(x)}{g(x)} = \lim_{x\to\infty}\frac{f(x)}{g(x)} + \lim_{x\to\infty}\frac{h(x)}{g(x)} = 1 + 0 = 1$$

so that $f + h \sim g$ and since h was an arbitrary function in $o(g)$ we conclude that if $f \sim g$ and $h = o(g)$ then $f + o(g) \sim g$.

8.

$$f(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n$$

$$\Rightarrow f(x) - a_0 - a_1(x - x_0) = (x - x_0)\sum_{n=2}^{\infty} a_n(x - x_0)^{n-1}$$

$$\Rightarrow \lim_{x\to x_0}\frac{f(x) - a_0 - a_1(x - x_0)}{x - x_0} = \lim_{x\to x_0}\sum_{n=2}^{\infty} a_n(x - x_0)^{n-1};$$

since the function defined by this series is continuous we may exchange the limit and the infinite sum to get

$$\sum_{n=2}^{\infty} a_n \lim_{x\to x_0}(x_0 - x_0)^{n-1} = 0.$$

Comment It follows that we might write $f(x) = a_0 + a_1(x - x_0) + o(x - x_0)$.

9. Suppose that $l(x) = a + b(x - x_0)$. Then $\lim_{x\to x_0}\frac{f(x)-l(x)}{x-x_0}$

$$= \lim_{x\to x_0}\frac{(a_0 - a) + (a_1 - b)(x - x_0)}{x - x_0} + \lim_{x\to x_0}\frac{\sum_{n=2}^{\infty} a_n(x - x_0)^n}{x - x_0};$$

now, as in Question 8, the second limit is equal to 0 while the first limit is

$$\lim_{x \to x_0} \frac{a_0 - a}{x - x_0} + (a_1 - b).$$

Now the first limit is undefined unless $a = a_0$, in which case it is 0 (and $l(x_0) = f(x_0) = a_0$) while the second limit is 0 if and only if $b = a_1$, from which the result follows from these observations and Question 8.

Comment Note that $\lim_{x \to x_0} \frac{f(x)}{x - x_0} = \lim_{x - x_0 \to 0} \frac{f((x_0 + (x - x_0))}{x - x_0} = f'(x_0) = \lim_{x \to x_0} \frac{l(x)}{x - x_0} = b$ so that the the tangent line to the graph of $f(x)$ at $x = x_0$ is the unique linear function $l(x)$ that matches both the value of the function and it derivative at the given value $x = x_0$. We can thus write $f(x) = l(x) + o(x - x_0)$ as $x \to x_0$.

10.

$$(n + O(n^{\frac{1}{2}})(n + O(\log n))^2$$

$$= n^3 + n^2(O(n^{\frac{1}{2}}) + 2O(\log n)) + n((2O(n^{\frac{1}{2}})O(\log n) + O^2(\log n)) + O(n^{\frac{1}{2}})O^2(\log n)$$

$$= n^3 + O(n^{\frac{5}{2}}),$$

where we have used considerations such that $n^2 O(n^{\frac{1}{2}}) = O(n^{\frac{5}{2}})$ using the rule $O(f)o(g) = o(fg)$; $2O(\log n) = o(n^\alpha)$ for any $0 < \alpha$ so that $n^2 \cdot 2O(\log n) = n^2 o(n^\alpha) = o(n^{2+\alpha})$, which can then be absorbed into the $O(n^{\frac{5}{2}})$ term by taking $\alpha \leq \frac{1}{2}$, and so forth.

Problem Set 5

1.

$$g'(x) = \lim_{\Delta x \to 0} \frac{1}{\Delta x} \Big(\int_a^b f(x + \Delta x, t)\, dt - \int_a^b f(x, t)\, dt \Big)$$

$$= \lim_{\Delta x \to 0} \int_a^b \frac{f(x + \Delta x, t) - f(x, t)}{\Delta x}\, dt = \lim_{\Delta x \to 0} \int_a^b \frac{f(x, t) + \frac{\partial f(x,t)}{\partial x} \Delta x + o(\Delta x) - f(x, t)}{\Delta x}\, dt$$

$$= \lim_{\Delta x \to 0} \int_a^b \Big(\frac{\partial f(x, t)}{\partial x} + \frac{o(\Delta x)}{\Delta x} \Big)\, dt = \int_a^b \frac{\partial f(x, t)}{\partial x}\, dt + \lim_{\Delta x \to 0} \int_a^b \frac{o(\Delta x)}{\Delta x}\, dt;$$

but as $\Delta x \to 0$ the integrand approaches 0 independently of t, this latter limit is 0 and therefore we may *differentiate inside the integral*:

$$g'(x) = \int_a^b \frac{\partial f(x, t)}{\partial x}\, dt.$$

Comment It may be shown by similar arguments that more generally for $g(x) = \int_{a(x)}^{b(x)} f(x, t)\, dt$ we get

$$g'(x) = f(x, b(x))b'(x) - f(x, a(x))a'(x) + \int_{a(x)}^{b(x)} \frac{\partial f(x, t)}{\partial x}\, dt.$$

2.

$$f(b) = \int_0^1 \frac{x^b - 1}{\log x}\,dx \Rightarrow f'(b) = \int_0^1 \frac{(\log x)x^b}{\log x}\,dx = \int_0^1 x^b\,dx =$$

$$\frac{x^{b+1}}{b+1}|_0^{x=1} = \frac{1}{b+1}.$$

$$f(b) = \int \frac{1}{b+1}\,db = \log(b+1) + C.$$

3. Putting $b = 0$ gives

$$\int_0^1 \frac{x^0 - 1}{\log x}\,dx = \log 1 + C \Rightarrow C = 0 \text{ and so } f(b) = \log(b+1).$$

4. Putting $b = 2$ now gives

$$\int_0^1 \frac{x^2 - 1}{\log x}\,dx = \log 3.$$

5. First

$$g(t) = \int_0^\infty e^{-tx}\,dx = [-\frac{1}{t}e^{-tx}]_{x=0}^\infty = [0 - -\frac{1}{t}] = \frac{1}{t}.$$

Since $g(t) = t^{-1}$ it follows that

$$g^{(n)}(t) = (-1)^n n! t^{-n-1} \tag{1}$$

for any $n \geq 0$. On the other hand, n-fold differentiation through the integral gives

$$g^{(n)}(t) = \int_0^\infty \frac{d^n(e^{-tx})}{dt}\,dx = \int_0^\infty (-1)^n x^n e^{-tx}\,dx \tag{2}$$

Equating (2) and (1) and putting $t = 1$ now gives

$$\int_0^\infty x^n e^{-x}\,dx = n!$$

6.

$$f(b) = \int_0^\infty \frac{\sin x}{x} e^{-bx}\,dx \Rightarrow f'(b) = -\int_0^\infty \sin x\, e^{-bx}\,dx.$$

Let $I = \int \sin x\, e^{-bx}\,dx$. Integrating by parts once gives $I = -\frac{\sin x}{b}e^{-bx} - \frac{1}{b}\int \cos x\, e^{-bx}\,dx$, and a second time gives

$$I = -\frac{\sin x}{b}e^{-bx} - \frac{1}{b}\big(-\frac{1}{b}\cos x\, e^{-bx} + \frac{1}{b}\int \sin x\, e^{-bx}\,dx\big)$$

$$\Rightarrow (1 + \frac{1}{b^2})I = -\frac{\sin x}{b}e^{-bx} + \frac{1}{b^2}\cos x\, e^{-bx}$$

$$\Rightarrow I = -\frac{b}{b^2+1}\sin x\, e^{-bx} + \frac{1}{b^2+1}\cos x\, e^{-bx}$$

$$\Rightarrow I = -\frac{b\sin x + \cos x}{b^2+1}e^{-bx}. \text{ Hence}$$

$$f'(b) = [\frac{b\sin x + \cos x}{b^2+1}e^{-bx}]_0^\infty = 0 - \frac{1}{b^2+1} = -\frac{1}{b^2+1}$$

$$\Rightarrow f(b) = -\arctan b + C.$$

7. Let $b \to \infty$ in Question 6 gives $0 = \frac{\pi}{2} + C$ so that $C = \frac{\pi}{2}$.

8. Since $\frac{\sin x}{x}$ is even we have upon putting $b = 0$

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx = 2\int_0^\infty \frac{\sin x}{x}\, dx = 2(\frac{\pi}{2} - \arctan 0) = \pi.$$

9. Let $f(x) = \cos x - 1 + \frac{x^2}{2}$. Then $f(0) = 1 - 1 + 0 = 0$. Also $f'(x) = x - \sin x > 0$ for all $x > 0$. Hence $f(x) \geq f(0) = 0$ for all $x \geq 0$.

Comment A rigorous justification of this final claim follows by applying the *Mean Value Theorem*, which will be one of the main topics in MA205 *Real Analysis*.

10. No. By Question 9 we have that for $x \geq 0$, $\cos x \geq 1 - \frac{x^2}{2}$ so that for $0 < \varepsilon < 1$ we have

$$\int_\varepsilon^1 \frac{\cos x}{x}\, dx \geq \int_\varepsilon^1 (\frac{1}{x} - \frac{x}{2})\, dx = [\ln x - \frac{x^2}{4}]_\varepsilon^1$$

$$= (0 - \frac{1}{4}) - (\ln\varepsilon - \frac{\varepsilon^2}{4}) = \frac{\varepsilon^2 - 1}{4} - \ln\varepsilon$$

and as $\varepsilon \to 0$ the final term approaches ∞.

Problem Set 6

1.

$$\Gamma(t+1) = \int_0^\infty x^t e^{-x}\, dx = -x^t e^{-x}|_0^\infty + \int_0^\infty t x^{t-1} e^{-x}\, dx$$

$$= -[0 - 0] + t\int_0^\infty x^{t-1}e^{-x}\, dx = t\Gamma(t).$$

In particular $\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-1) = \cdots = n(n-1)\cdots 2\Gamma(1)$. Now $\Gamma(1) = \int_0^\infty e^{-x}\, dx = -e^{-x}|_0^\infty = -[0-1] = 1$. Hence $\Gamma(n+1) = n!$.

Comment See Question 5 on Set 5 for an alternative calculation.

2.

$$\Gamma(\frac{1}{2}) = \int_0^\infty \frac{e^{-x}}{\sqrt{x}}\, dx, \text{ put } u = \sqrt{x} \text{ so that } du = \tfrac{dx}{2u}$$

$$\Rightarrow \Gamma(\frac{1}{2}) = 2\int_0^\infty e^{-u^2}\,du,$$

the latter is the Gaussian integral with value $\frac{1}{2}\sqrt{\pi}$, thus giving that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

3. The $n = 0$ case follows from Question 2. Next we integrate by parts as follows:

$$\Gamma(\frac{1}{2}+n) = \int_0^\infty x^{n-\frac{1}{2}}e^{-x}\,dx \text{ put } u = x^{n-\frac{1}{2}},\ dv = e^{-x}dx$$

$$= -x^{n-\frac{1}{2}}e^{-x}|_0^\infty + (n-\frac{1}{2})\int_0^\infty x^{n-\frac{3}{2}}e^{-x}\,dx = \frac{2n-1}{2}\Gamma(n-\frac{1}{2}),\text{ which by induction equals}$$

$$\frac{2n-1}{2}\cdot\frac{(2(n-1))!}{4^{n-1}(n-1)!}\sqrt{\pi} = \frac{(2n)!}{4^n n!}\sqrt{\pi}.$$

4. The $n = 0$ case again follows from Question 2. Next we integrate by parts as follows:

$$\Gamma(\frac{1}{2}-n) = \int_0^\infty x^{-n-\frac{1}{2}}e^{-x}\,dx \text{ put } u = e^{-x},\ dv = x^{-n-\frac{1}{2}}dx \Rightarrow v = \tfrac{2}{1-2n}x^{-n+\frac{1}{2}}$$

$$= \frac{2}{1-2n}x^{-n+\frac{1}{2}}e^{-x}|_0^\infty - \frac{2}{2n-1}\int_0^\infty x^{-n+\frac{1}{2}}e^{-x}\,dx = \frac{2}{2n-1}\Gamma(\frac{1}{2}-(n-1)),\text{ which by induction equals}$$

$$\frac{2}{2n-1}\cdot\frac{(-4)^{n-1}(n-1)!}{(2(n-1))!}\sqrt{\pi} = \frac{(-4)^n n!}{(2n)!}\sqrt{\pi}.$$

5.

$$\Gamma(x)\Gamma(y) = \int_0^\infty\int_0^\infty e^{-u-v}u^{x-1}v^{y-1}\,dudv.$$

6.

$$J = \left|\begin{array}{cc} \frac{\partial u}{\partial z} & \frac{\partial u}{\partial t} \\ \frac{\partial v}{\partial t} & \frac{\partial v}{\partial t} \end{array}\right| = \left|\begin{array}{cc} t & z \\ 1-t & -z \end{array}\right| = |-tz - -z(1-t)| = |-tz - z + tz| = z.$$

Next we note that $u+v = zt+z(1-t) = z$, so that the range of z is $0 \le z \le \infty$. On the other hand $t = \frac{u}{z} = \frac{u}{u+v}$ so that $0 \le t \le 1$.

7. From Question 6 we get through the given substitutions:

$$\Gamma(x)\Gamma(y) = \int_0^\infty\int_0^1 e^{-z}(zt)^{x-1}(z(1-t))^{y-1}z\,dtdz$$

$$= \Big(\int_0^\infty e^{-z}z^{x+y-1}\,dz\Big)\Big(\int_0^1 t^{x-1}(1-t)^{y-1}\,dt\Big) = \Gamma(x+y)B(x,y),$$

$$\therefore B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

8. Substitute $u = \frac{t}{1-t}$ in the beta integral. Hence range of u is $(0, \infty)$. We have $u - ut - t = 0$ so that $t = \frac{u}{u+1}$, $dt = \frac{du}{(u+1)^2}$ and $1 - t = 1 - \frac{u}{u+1} = \frac{1}{u+1}$. Hence we obtain:

$$B(x,y) = \int_0^\infty (\frac{u}{u+1})^{x-1}(\frac{1}{u+1})^{y-1}\frac{du}{(u+1)^2} = \int_0^\infty \frac{u^{x-1}}{(u+1)^{x+y}}du.$$

9. Questions 7 and then 1 allow us to write:

$$B(x,y+1) + B(x+1,y) = \frac{\Gamma(x)\Gamma(y+1) + \Gamma(x+1)\Gamma(y)}{\Gamma(x+y+1)}$$

$$= \frac{y\Gamma(x)\Gamma(y) + x\Gamma(x)\Gamma(y)}{(x+y)\Gamma(x+y)} = \frac{(x+y)(\Gamma(x)+\Gamma(y))}{(x+y)\Gamma(x+y)}$$

$$= \frac{\Gamma(x)+\Gamma(y)}{\Gamma(x+y)} = B(x,y).$$

10. Similarly

$$B(x+1,y) = \frac{\Gamma(x+1)\Gamma(y)}{\Gamma(x+y+1)} = \frac{x\Gamma(x)\Gamma(y)}{(x+y)\Gamma(x+y)} = B(x,y)\cdot\frac{x}{x+y}.$$

Problem Set 7

1. $T_0(\cos\theta) = \cos 0 = 1$, so that $T_0(x) = 1$. $T_1(\cos\theta) = \cos\theta \Rightarrow T_1(x) = x$.
2. $T_2(\cos\theta) = \cos 2\theta = 2\cos^2\theta - 1 \Rightarrow T_2(x) = 2x^2 - 1$.

$$T_3(\cos\theta) = \cos 3\theta = \cos\theta\cos 2\theta - \sin\theta\sin 2\theta = \cos\theta\cos 2\theta - 2\cos\theta\sin^2\theta$$

$$= x(2x^2-1) - 2x(1-x^2) = 2x^3 - x - 2x + 2x^3 = 4x^3 - 3x = x(4x^2-3).$$

3. The given identity can be written as

$$\cos(n+1)\theta = 2\cos\theta\cos n\theta - \cos(n-1)\theta \Leftrightarrow \cos(n+1)\theta + \cos(n-1)\theta = 2\cos\theta\cos n\theta.$$

Now applying the general identity $\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$ to the left hand side of the previous equation gives:

$$2\cos\frac{n\theta+\theta+n\theta-\theta}{2}\cos\frac{n\theta+\theta-n\theta+\theta}{2} = 2\cos\theta\cos n\theta$$

in agreement with the right hand side, as we require.

4. Hence

$$T_4(x) = 2xT_3(x) - T_2(x) = 2x(4x^3-3x) - (2x^2-1) = 8x^4 - 6x^2 - 2x^2 + 1 = 8x^4 - 8x^2 + 1.$$

5.

$$T_n(T_m(\cos\theta)) = T_n(\cos m\theta) = \cos n(m\theta) = \cos nm\theta = T_{nm}(\cos\theta)$$

and so $T_n(T_m(x)) = T_{nm}(x)$.

6.

$$2U_1(x) = T_2^{'}(x) = (2x^2 - 1)' = 4x \Rightarrow U_1(x) = \frac{4x}{2} = 2x.$$

$$3U_2(x) = T_3'(x) = (4x^3 - 3x)' = 12x^2 - 3 \Rightarrow U_3(x) = 4x^2 - 1.$$

$$4U_3(x) = T_4'(x) = (8x^4 - 8x^2 + 1)' = 32x^3 - 8x \Rightarrow U_3(x) = 8x^3 - 4x = 4x(2x^2 - 1).$$

7. We have

$$x = \cos\theta \Rightarrow \frac{dx}{d\theta} = -\sin\theta \Rightarrow \frac{d\theta}{dx} = -\frac{1}{\sin\theta}. \text{ Hence}$$

$$T_n'(x) = -\frac{d(\cos n\theta)}{d\theta} \cdot \frac{1}{\sin\theta} = \frac{n\sin n\theta}{\sin\theta}$$

$$\Rightarrow U_{n-1}(x) = \frac{1}{n}T_n'(x) = \frac{\sin n\theta}{\sin\theta}.$$

8. Using Question 7 we have

$$U_n(x) - xU_{n-1}(x) = \frac{\sin(n+1)\theta}{\sin\theta} - \frac{\cos\theta\sin n\theta}{\sin\theta} = \frac{\sin\theta\cos n\theta + \cos\theta\sin n\theta - \cos\theta\sin n\theta}{\sin\theta}$$

$$= \cos n\theta = T_n(x).$$

9. Again using Question 7:

$$U_n(x) - U_{n-2}(x) = \frac{\sin(n+1)\theta}{\sin\theta} - \frac{\sin(n-1)\theta}{\sin\theta} =$$

$$\frac{\sin\theta\cos n\theta + \cos\theta\sin n\theta - (\sin n\theta\cos\theta - \cos n\theta\sin\theta)}{\sin\theta} =$$

$$\frac{2\sin\theta\cos n\theta}{\sin\theta} = 2\cos n\theta = 2T_n(x).$$

10.

$$T_n''(x) = -\frac{n^2\cos n\theta\sin\theta - n\cos\theta\sin n\theta}{\sin^2\theta} \cdot \frac{1}{\sin\theta} = \frac{n\cos\theta\sin n\theta - n^2\sin\theta\cos n\theta}{\sin^3\theta}$$

$$\Rightarrow (1-x^2)y^{''}(x) = \sin^2\theta T_n''(x) = \frac{n\cos\theta\sin n\theta - n^2\sin\theta\cos n\theta}{\sin\theta};$$

$$xy' = \frac{n\cos\theta\sin n\theta}{\sin\theta},\ n^2y = n^2\cos n\theta.$$

Summing the terms gives

$$(1-x^2)y'' - xy' + n^2y = \frac{n\cos\theta\sin n\theta - n^2\sin\theta\cos n\theta - n\cos\theta\sin n\theta + n^2\sin\theta\cos n\theta}{\sin\theta} = 0.$$

Problem Set 8

1.
$$\mathcal{L}\{1\} = \int_0^\infty e^{-st}dt = -\frac{1}{s}[e^{-st}]_0^\infty = -\frac{1}{s}[0-1] = \frac{1}{s}.$$

2. The $n = 0$ case that anchors the induction is Question 1, so suppose that $n \geq 0$ and consider $\mathcal{L}\{t^{n+1}\} = \int_0^\infty e^{-st}t^{n+1}dt$. Integrating by parts with $u = t^{n+1} \Rightarrow du = (n+1)t^n$, $dv = e^{-st}dt \Rightarrow v = -\frac{1}{s}e^{-st}$ we obtain

$$\mathcal{L}\{t^{n+1}\} = -\frac{t^{n+1}}{s}e^{-st}\Big|_0^\infty + \frac{n+1}{s}\int_0^\infty t^n e^{-st}dt$$

which by induction is equal to

$$(0-(-0)) + \frac{n+1}{s}\mathcal{L}\{t^n\} = \frac{n+1}{s}\cdot\frac{n}{s^{n+1}} = \frac{(n+1)!}{s^{n+2}}$$

and the induction continues, thus showing that for all positive s,

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}\,\forall\, n \geq 0.$$

3. For $s > a$ we obtain:

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st}e^{at}dt = \int_0^\infty e^{(a-s)t}dt = \frac{1}{a-s}e^{(a-s)t}\Big|_0^\infty = 0 - \frac{1}{a-s} = \frac{1}{s-a}.$$

4. Integrating by parts with $du = e^{-st} \Rightarrow u = -\frac{1}{s}e^{-st}dt$ and $v = \sin at\,dt \Rightarrow dv = a\cos at$ gives

$$\mathcal{L}\{\sin at\} = \int_0^\infty e^{-st}\sin at\,dt = -\frac{1}{s}e^{-st}\sin at\Big|_0^\infty + \frac{a}{s}\int_0^\infty e^{-st}\cos at\,dt$$
$$= \frac{a}{s}\int_0^\infty e^{-st}\cos at\,dt.$$

Integrating by parts a second time with $du = e^{-st} \Rightarrow u = -\frac{1}{s}e^{-st}$ and $v = \cos at\,dt \Rightarrow dv = -a\sin at$ gives

$$\mathcal{L}\{\sin a\} = \frac{a}{s}\big(-\frac{1}{s}e^{-st}\cos at\Big|_0^\infty - \frac{a}{s}\int_0^\infty e^{-st}\sin t\,dt$$

$$\Rightarrow (1+\frac{a^2}{s^2})\mathcal{L}\{\sin at\} = \frac{a}{s^2} \Rightarrow \mathcal{L}\{\sin at\} = \frac{s^2}{a^2+s^2}\cdot\frac{a}{s^2} = \frac{a}{a^2+s^2},\ s > 0.$$

5.

$$\mathcal{L}\{(af(t)+bg(t)\} = \int_0^\infty e^{-st}(af(t)+bg(t))dt = a\int_0^\infty e^{-st}f(t)dt + b\int_0^\infty e^{-st}bg(t)dt$$

$$= a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}.$$

6. Again, we integrate by parts with $u = e^{-st} \Rightarrow du = -se^{-st}\,dt$ and $f'(t) = \frac{dv}{dt} \Rightarrow f(t) = v$, to obtain

$$\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t)dt = e^{-st} f(t)\Big|_0^\infty + s\int e^{-st} f(t)dt$$

$$\Rightarrow \mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0).$$

7. Applying Question 6 to $f'(t)$ we obtain:

$$\mathcal{L}\{f''(t)\} = s\mathcal{L}\{f'(t)\} - f'(0) = s(s\mathcal{L}\{f(t)\} - f(0)\} - f'(0)$$

$$= s(s\mathcal{L}\{f(t)\} - f(0)\} - f'(0) = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0).$$

Comment A natural inductive argument now gives that

$$\mathcal{L}\{f^{(n)}(t)\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0).$$

8. Taking the Laplace transform of both sides gives $\mathcal{L}\{y'' - y' - 2y\} = \mathcal{L}\{y''\} - \mathcal{L}\{y'\} - 2\mathcal{L}\{y\} = \mathcal{L}\{0\} = 0$, hence using Questions 6 and 7 and write $Y = Y(s)$ for $\mathcal{L}\{f(t)\}$we have:

$$s^2Y - sy(0) - y'(0) - sY + y(0) - 2Y = 0 \Rightarrow Y(s^2 - s - 2) = (s-1)y(0) + y'(0)$$

$$\Rightarrow Y = \frac{(s-1)(1)+0}{s^2-s-2} = \frac{s-1}{s^2-s-2}.$$

9. Factorizing the denominator and using the 'Cover-up' method gives:

$$\frac{s-1}{s^2-s-2} = \frac{s-1}{(s-2)(s+1)} = \frac{(2-1)/(2+1)}{s-2} = \frac{(-1-1)/(-1-2)}{s+1} = \frac{1/3}{s-2} + \frac{2/3}{s+1}.$$

10. We take the inverse and use the result of Question 5 in particular to obtain:

$$y(t) = \mathcal{L}^{-1}\{\frac{1/3}{s-2} + \frac{2/3}{s+1}\} = \frac{1}{3}\mathcal{L}^{-1}\{\frac{1}{s-2}\} + \frac{2}{3}\mathcal{L}^{-1}\{\frac{1}{s+1}\}$$

$$\Rightarrow y(t) = \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t}.$$

Problem Set 9

1. As x^2 is even, we get a cosine series only with $\pi a_n = 2\int_0^\pi x^2 \cos nx dx$. For $n = 0$ we get $2[\frac{1}{3}x^3]_0^\pi = \frac{2}{3}\pi^3 \Rightarrow a_0 = \frac{2\pi^2}{3}$. Otherwise, integrating by parts twice we obtain:

$$a_n = \frac{2x^2}{n}\sin nx\Big|_0^\pi - \frac{4}{n}\int_0^\pi x\sin nx dx = -\frac{4}{n}\Big(-\frac{x}{n}\cos nx\Big|_0^\pi + \frac{1}{n}\int_0^\pi \cos nx dx\Big)$$

$$= \frac{4(-1)^n\pi}{n^2}, \text{ and so}$$

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx.$$

2. Put $x = 0$ in the series of Question 1 gives:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12} \Rightarrow 1 - \frac{1}{2^2} + \frac{1}{3^2} - \cdots = \frac{\pi^2}{12}.$$

3. Put $x = \pi$ in the series of Question 1: now $\cos n\pi = \pm 1$ according as n is even or odd. Hence the term $(-1)^n \cos n\pi = 1$ for all n and we obtain:

$$\pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{1}{n^2} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \pi^2 (\frac{1}{4} - \frac{1}{12}) = \frac{\pi^2}{6}.$$

4. Again, we get only a cosine series with $\pi a_n = 2\int_0^\pi x \cos nx\, dx$. For $n = 0$ we get $\pi a_0 = 2\frac{\pi^2}{2}$ so that $a_0 = \pi$. For $n \neq 0$ we integrate by parts and obtain

$$\int_0^\pi x \cos nx\, dx = \frac{1}{n} x \sin nx|_0^\pi - \frac{1}{n}\int_0^\pi \sin nx\, dx =$$

$$-\frac{2}{n^2}\ n \text{ odd}$$

and the integral is 0 for even positive integers. Consequently

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^2}.$$

Finally we put $x = 0$ and obtain

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}.$$

5. Again since $f(x)$ is even, we will get a cosine series only. We have $\pi a_0 = 2\int_0^\pi \cos \mu x\, dx = \frac{2}{\mu}[\sin \mu x]_0^\pi = \frac{2}{\mu} \sin \mu\pi$.

$$a_n = \frac{2}{\pi}\int_0^\pi \cos \mu x \cos nx\, dx = \frac{1}{\pi}\int_0^\pi (\cos((\mu+n)x) - \cos((\mu-n)x)\, dx$$

$$= \frac{\sin((\mu+n)\pi)}{(\mu+n)\pi} + \frac{\sin((\mu-n)\pi)}{\mu - n} =$$

$$= \frac{\sin \mu\pi \cos n\pi(\mu - n) + \sin \mu\pi \cos n\pi(\mu+n)}{(\mu+n)(\mu-n)\pi}$$

$$= \frac{2\mu(-1)^n}{\pi(\mu^2 - n^2)} \sin \mu\pi.$$

Combining these results gives

$$\cos\mu x = \frac{\sin\mu\pi}{\mu\pi} + \sum_{n=1}^{\infty} \frac{2\mu(-1)^n \sin\mu x}{\pi(\mu^2 - n^2)} \cdot \cos nx$$

$$\Rightarrow \cos\mu x = \frac{2\mu\sin\mu\pi}{\pi}\left(\frac{1}{2\mu^2} - \frac{\cos x}{\mu^2 - 1^2} + \frac{\cos 2x}{\mu^2 - 2^2} - \cdots\right).$$

6. We now put $x = \pi$, divide both sides of the final equation in Question 6 by $\sin\mu\pi$ and write the symobl x instead of μ to obtain:

$$\cot\pi x = \frac{2x}{\pi}\left(\frac{1}{2x^2} + \frac{1}{x^2 - 1^2} + \frac{1}{x^2 - 2^2} + \frac{1}{x^2 - 3^2} + \cdots\right)$$

$$\Rightarrow \cot\pi x - \frac{1}{\pi x} = -\frac{2x}{\pi}\left(\frac{1}{1^2 - x^2} + \frac{1}{2^2 - x^2} + \frac{1}{3^2 - x^2} + \cdots\right).$$

7. For $n = 0$ both sides of the identity return $\frac{1}{2} = \frac{1}{2}$ ($u \neq n\pi$ so denominator is well-defined), thereby anchoring the induction. Now let $n \geq 1$. By induction the expression on the left may be replaced by

$$\frac{\sin(n - \frac{1}{2})u}{2\sin\frac{u}{2}} + \cos nu = \frac{\sin(n - \frac{1}{2})u + 2\sin\frac{u}{2}\cos nu}{2\sin\frac{u}{2}}$$

$$= \frac{\sin nu\cos\frac{u}{2} - \cos nu\sin\frac{u}{2} + 2\sin\frac{u}{2}\cos nu}{2\sin\frac{u}{2}} =$$

$$\frac{\sin nu\cos\frac{u}{2} + \cos nu\sin\frac{u}{2}}{2\sin\frac{u}{2}} = \frac{\sin(n + \frac{1}{2})u}{2\sin\frac{u}{2}},$$

and the induction continues.

8. Putting $z = e^{iu}$ in the standard geometric series we obtain:

$$1 + e^{iu} + e^{2iu} + \cdots + e^{niu} = \frac{1 - e^{i(n+1)u}}{1 - e^{iu}}.$$

The real part of the LHS is $1 + \cos u + \cos 2u + \cdots + \cos nu$, while multiplying the RHS top and bottom by $e^{-\frac{1}{2}iu}$ gives:

$$\frac{e^{-\frac{1}{2}iu} - e^{i(n+\frac{1}{2})u}}{e^{-\frac{1}{2}iu} - e^{\frac{1}{2}iu}} = i\frac{e^{-\frac{1}{2}iu} - e^{-i(n+\frac{1}{2})u}}{2\sin\frac{u}{2}};$$

taking the real part of this expression we obtain

$$\frac{\sin\frac{u}{2} + \sin(n + \frac{1}{2})u}{2\sin\frac{u}{2}} = \frac{1}{2} + \frac{\sin(n + \frac{1}{2})u}{2\sin\frac{u}{2}}$$

from whence the Lagrange identity follows.

9. Taking the sum suggested we obtain $f(t) + ig(t) =$

$$1 + ae^{it} + a^2e^{i2t} + a^3e^{i3t} + \cdots = \frac{1}{1 - ae^{it}}, \text{ (as } |a| < 1).$$

10. The RHS of the answer to Question 9 is:

$$= \frac{1}{1 - a\cos t - ia\sin t} = \frac{(1 - a\cos t) + ia\sin t}{(1 - a\cos t)^2 + a^2\sin^2 t} = \frac{1 - a\cos t + ia\sin t}{1 - 2a\cos t + a^2};$$

the real part of this expression is $f(t)$ thereby giving,

$$\sum_{n=0}^{\infty} a^n \cos nt = \frac{1 - a\cos t}{1 - 2a\cos t + a^2}, \ (-1 < a < 1).$$

Problem Set 10

1. We use x as the t parameter in that we put $x(t) = t, y(t) = t^2$ for $-1 \leq t \leq 1$. We thus get $\dot{x}(t) = 1$, $\dot{y}(t) = 2t$ and so

$$\int_C x\, ds = \int_{-1}^{1} t(1 + 4t^2)^{\frac{1}{2}}\, dt = [\frac{(1 + 4t^2)^{\frac{3}{2}}}{12}]_{-1}^{1} = 0.$$

2. We parametrize C as $x(t) = \cos t$, $y(t) = \sin t$ with $0 \leq t \leq \frac{\pi}{2}$ so that $\dot{x}(t) = -\sin t$, $\dot{y}(t) = \cos t$. We then get for our integral I:

$$I = \int_0^{\frac{\pi}{2}} \cos t \sin t \sqrt{(\cos t)^2 + (-\sin t)^2}\, dt = \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin 2t \cdot 1\, dt$$

$$= -\frac{1}{4}[\cos 2t]_0^{\frac{\pi}{2}} = -\frac{1}{4}[-1 - 1] = \frac{1}{2}.$$

3. The curve C has two parts, parametrized as follows:

$$C_1 : r(t) = (2\cos t, 2\sin t),\ (0 \leq t \leq \pi),\ C_2 : r(t) = (-2 + 4t, 0),\ 0 \leq t \leq 1.$$

$$C_1 : \dot{x}(t) = -2\sin t,\ \dot{y}(t) = 2\cos t,\ \sqrt{} = (4\sin^2 t + 4\cos^2 t)^{\frac{1}{2}} = 2.$$

Hence

$$\int_{C_1} f(x, y)\, ds = \int_0^{\pi} (2\cos t + 4\sin^2 t)2\, dt = 4\int_0^{\pi} (\cos t + 1 - \cos 2t)\, dt$$

$$= 4[\sin t + t - \frac{1}{2}\sin 2t]_0^{\pi} = 4[(0 + \pi - 0) - (0 + 0 - 0)] = 4\pi.$$

$$C_2 : \dot{x}(t) = 4,\ \dot{y}(t) = 0,\ \text{so} \surd = 4 \text{ and}$$

$$\int_{C_2} f(x,y)\, ds = \int_0^1 (-2+4t+0^2)4\, dt = 8\int_0^1 (2t-1)\, dt$$

$$= 8[t^2 - t]_0^1 = 8[(1-1)-(0-0)] = 0.$$

$$\therefore \int_C f(x,y)\, dx = \int_{C_1} f(x,y)\, ds + \int_{C_2} f(x,y)\, dx = 4\pi + 0 = 4\pi.$$

Comment Remember that the directed line segment between points with position vectors $\mathbf{a}$ and $\mathbf{b}$ can be parametrized as $\mathbf{r}(t) = \mathbf{b} + (1-t)(\mathbf{a} - \mathbf{b})$ $(0 \le t \le 1)$.

4. We have $\dot{x}(t) = -\sin t$, $\dot{y}(t) = \cos t$ and $\dot{z}(t) = 3$. Hence

$$\int_C f(x,y,z)\, ds = \int_a^b f(x(t), y(t), z(t))\sqrt{\dot{x}^2(t) + \dot{y}^2(t) + \dot{z}^2(t)}\, dt$$

$$= \int_0^{4\pi} \cos t \sin t(3t)\sqrt{(-\sin t)^2 + (\cos t)^2 + 3^2} dt = \sqrt{10}\int_0^{4\pi} \frac{3t}{2} \sin 2t\, dt.$$

Ignoring the constant multiplier of $\frac{3\sqrt{10}}{2}$ for the moment, we integreate by parts $(u = t, dv = \sin 2t$,so that $u' = 1$ and $v = -\frac{\cos 2t}{2})$ to obtain

$$[t(-\cos 2t)/2[_0^{4\pi} - \int_0^{4\pi} -\frac{\cos 2t}{2}\, dt$$

$$= [-\frac{t\cos 2t}{2} + \frac{\sin 2t}{4}]_0^{4\pi};$$

and putting back our constant factor reveals the full answer:

$$\frac{3\sqrt{10}}{2}[(-\frac{4\pi}{2} - 0) - (0-0)] = -3\sqrt{10}\pi.$$

5. The vector $\mathbf{T} = \frac{\mathbf{r}'(t)}{||\mathbf{r}'(t)||}$. Moreover $\frac{ds}{dt} = \sqrt{\dot{x}^2(t) + \dot{y}^2(t)} = ||\mathbf{r}(t)||$. Hence

$$\int_C \mathbf{F}\bullet d\mathbf{r} = \int_C \mathbf{F}\bullet \frac{d\mathbf{r}}{dt}\, dt = \int_C \mathbf{F} \bullet \mathbf{T}||\mathbf{r}(t)||\, dt = \int_C \mathbf{F} \bullet \mathbf{T}\, ds.$$

6.

$$\mathbf{F}(r(t)) = 8t^2(t^2)(t^3)\mathbf{i} + 5(t^3)\mathbf{j} - 4t(t^2)\mathbf{k} = 8t^7\mathbf{i} + 5t^3\mathbf{j} - 4t^3\mathbf{k}.$$

Next we obtain $\mathbf{r}'(t) = 1\mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k}$. Then we have $\int_C \mathbf{F}\bullet d\mathbf{r} = \int_a^b \mathbf{F}(r(t))\bullet \mathbf{r}'(t)\, dt$

$$= \int_{t=0}^1 (8t^7 \times 1) + (5t^3 \times 2t) + ((-4t^3) \times 3t^2)\, dt = \int_0^1 (8t^7 + 10t^4 - 12t^5)\, dt$$

$$= [t^8 + 2t^5 - 2t^6]_{t=0}^1 = (1+2-2) - (0+0-0) = 1.$$

7. We need to parametrize the line segment in the given direction:

$$\mathbf{r}(t) = (-1, 2, 0) + t((3, 0, 1) - (-1, 2, 0)\,\mathbf{r}'(t) = ((4, -2, 1).$$

$$\mathbf{F}(\mathbf{r}(t)) = (4t-1)t\mathbf{i} - (2-2t)t\mathbf{k} = (4t^2 - t)\mathbf{i} - (2t - 2t^2)\mathbf{k}$$

$$\int_C \mathbf{F}\bullet d\mathbf{r} = \int_{t=a}^{b} \mathbf{F}(\mathbf{r}(t)) \bullet \mathbf{r}'(t)\,dt$$

$$= \int_0^1 ((4t^2 - t) \times 4) + (0 \times (-2)) +) - (2t - 2t^2) \times 1)\,dt$$

$$= \int_0^1 (18t^2 - 6t)\,dt = [6t^3 - 3t^2]_0^1 = (6-3) - (0-0) = 3.$$

8. Using the parametrization of C by t we have by definition

$$I = \int_C P(x, y)\,dx = \int_a^b P(x(t), y(t))\frac{dx}{dt}\,dt.$$

We re-write the integral using the substitution $t = \alpha(u)$ so that $dt = \alpha'(u)du$, $b = \alpha^{-1}(d), a = \alpha^{-1}(c)$; we replace $\frac{dx}{dt}$ by $\frac{dx}{du}\frac{du}{dt} = \frac{dx}{du}\frac{1}{\alpha'(u)}$ so that by the Chain rule:

$$I = \int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)} P(x(\alpha(u), y(\alpha(u))\frac{dx}{du}\frac{1}{\alpha'(u)}\alpha'(u)du$$

$$= \int_c^d P(\tilde{x}(u), \tilde{y}(u))\,dx.$$

which is the line integral I now expressed in terms of the parametrization of C by u. The same calculation with the variable x replaced by y also holds, showing that the line integral of $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ has value that is independent of the parametrization of the curve of integration C.

9. Similarly let $I = \int_c f(x, y)\,ds$ so that, using the parametrization of C by t

$$I = \int_a^b f(x(t), y(t))\sqrt{\dot{x}^2(t) + \dot{y}^2(t)}dt.$$

Now $\dot{x}(t) = \frac{dx}{dt} = \frac{dx}{du}\frac{du}{dt} = \tilde{x}(u)\frac{1}{\alpha'(u)}$ and similarly $\frac{dy}{dt} = \tilde{y}(u)\frac{1}{\alpha'(u)}$. Hence making the substituion $t = \alpha(u)$ allows us to re-write I as

$$I = \int_c^d f(\tilde{x}(u), \tilde{y}(u))\sqrt{\frac{\tilde{x}^2(u) + \tilde{y}^2(u)}{(\alpha'(u))^2}}\alpha'(u)\,du = \int_c^d f(\tilde{x}(u), \tilde{y}(u))\sqrt{\tilde{x}^2(u) + \tilde{y}^2(u)}\,du$$

which is the line integral for the real-valued function $f(x, y)$ expressed in terms of the alternative parametrization of C.

10. We have $(x(t), y(t)) = (t+3, t^2 - t + 2) = (u, u^2 - 7u + 14) = (\tilde{x}(u), \tilde{y}(u))$ with $-3 \le t \le 2$. If this is two parametrizations of the same curve we must have

$u = t+3$ so inverting we get $t = u-3$ so that, in our notation $t = \alpha(u) = u-3$. We need to check that $y(t) = y(\alpha(u)) = \tilde{y}(u)$;

$$y(\alpha(u)) = y(u-3) = (u-3)^2 - (u-3) + 2 = u^2 - 7u + 14 = \tilde{y}(u)$$

as required. Also $\alpha'(u) = 1 > 0$ so passage from t to u does give another parametrization of our curve with endpoing $a = \alpha(c) \Rightarrow -3 = c-3$ so that $c = 0$ and $b = \alpha(d) \Rightarrow -2 = d-3$ so that $d = 1$. We note that $\tilde{x}(0), \tilde{y}(0)) = (0, 14)$ and $(\tilde{x}(1), \tilde{y}(1)) = (1, 8)$, which return the correct endpoints of C. Hence, by Question 8, the integral $\int_C \mathbf{F}\bullet d\mathbf{r}$ should yield the same result with both parametrizations. Now $\mathbf{r}'(t) = (1, 2t-1)$ and so

$$\int_C \mathbf{F}\bullet d\mathbf{r}(t) = \int_{-3}^{-2} (2(t+3)^2, -t^2+t-2)\bullet(1, 2t-1)\, dt = \int_{-3}^{-2} \big(2(t+3)^2 - (t^2-t+2)(2t-1)\big)\, dt$$

$$= \int_{-3}^{-2} (-2t^3 + 5t^2 + 7t + 20)\, dt = [-\frac{t^4}{2} + \frac{5t^3}{3} + \frac{7t^2}{2} + 20t]_{-3}^{-2}$$

$$= [(-8 - \frac{40}{3} + 14 - 40) - (-\frac{81}{2} - 45 + \frac{63}{2} - 60)] = 66\frac{2}{3}.$$

On the other hand, using the alternative parametrization $\tilde{\mathbf{r}}'(u) = (1, 2u-7)$ and so

$$\int_C \mathbf{F}\bullet d\mathbf{r}(u) = \int_0^1 (2u^2, -u^2 + 7u - 14) \bullet (1, 2u-7)\, du$$

$$= \int_0^1 (2u^2 + (2u-7)(-u^2 + 7u - 14)\, du = \int_0^1 (-2u^3 + 23u^2 - 77u + 98)\, du$$

$$= [-\frac{u^4}{2} + \frac{23u^2}{3} - \frac{77u^2}{2} + 98u]_0^1 = (-\frac{1}{2} + \frac{23}{2} - \frac{77}{2} + 98) - (0) = 66\frac{2}{3}.$$